Lecture 19 - Introduction to Graph Signal Processing

ECE 5260 – ORIE 5735

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Cornell Tech

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Content

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Outline

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'Big Data" from network processes

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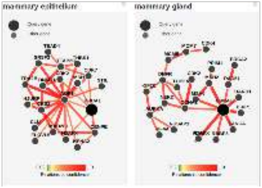

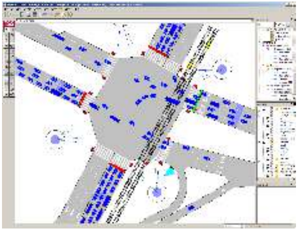

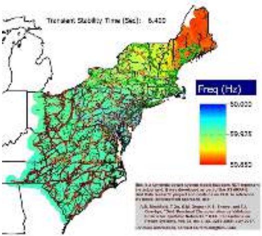

Network dynamics produce data (Module 2)

- Processes affected by **relational** parameters: social, economic, biological.
- Processes due to **networked** infrastructures: transportation, gas, water, electric power

From shallow embeddings to GSP

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- In discussing network embeddings and GRL last time, we represented each node by a low-dimensional vector derived from graph structure;
  - E.g. in spectral methods, those coordinates come from leading or trailing eigenvectors of a graph matrix such as the Laplacian.

  Essentially, we refined what we discussed in Module 1 as nodal characteristics (centrality etc.).
- We also mentioned that nodal labels can be used in (semi) supervised GRL. These would be **graph signals**.

Graph Signal Processing (GSP)

GSP is a generalization Digital Signal Processing (DSP) and Fourier analysis for such signals.

Two views of the same spectrum

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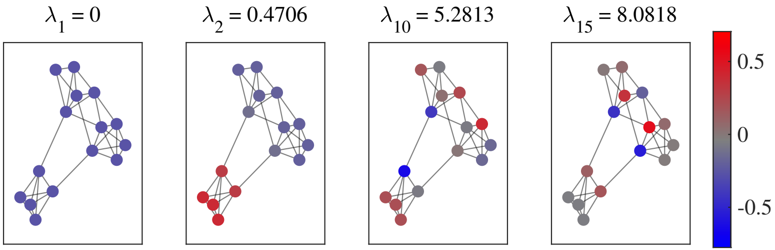
Spoiler alert

The natural Fourier basis for GSP is given by the eigenvector of the Graph Laplacian.

- **Embedding view**: use a few (least significant) eigenvectors to assign coordinates to nodes.
- **GSP view**: use all eigenvectors as a basis to analyze signals on the graph. *Low eigenvalue* $\Leftrightarrow$ *low graph frequency.*



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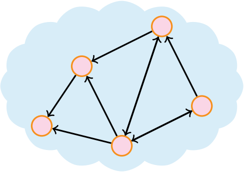
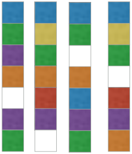
Graph Signal Processing Modeling

- Associate a *graph* $G = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ to the coordinate system of the data set
- Graph signals are vectors defined on nodes of G, i.e., $\boldsymbol{s} \in \mathbb{R}^N$.
- Generalize DSP extending the notion of shift operator $z^{-1} \rightarrow \ \mathbf{S} = \mathbf{\Phi}(\mathbf{A})$



Filter & Generate $\longrightarrow$



GSP tools

Various DSP techniques have been extended to graphs — graph Fourier analysis, interpolation, sampling, multi-scale filtering,

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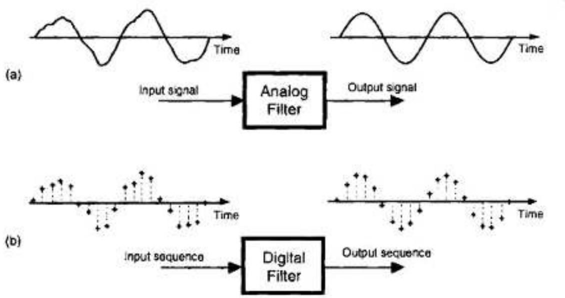
The notion of filter



- Digital Signal Processing (DSP) focuses on time series $x_t, \ t \in \mathbb{Z}$ and linear operators called filters
- The input of the filter is a time series $s_t \ t \in \mathbb{Z}$ and the output is a linear combination of the inputs

The notion of filter

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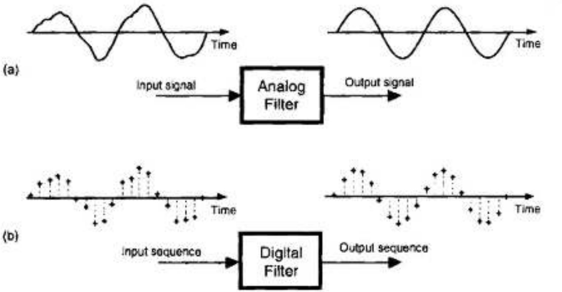

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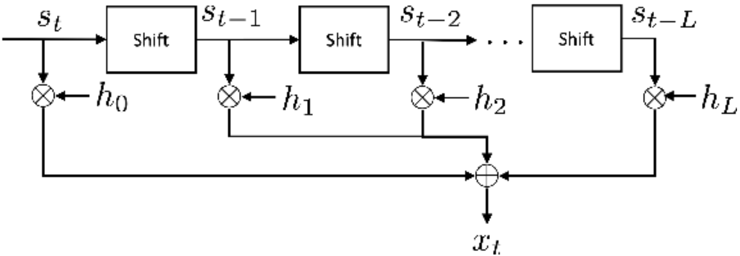

- For a Linear Time Invariant (LTI) the **input-output** relationship of a *finite memory* and *causal filter*

$$x_t = h_0 s_t + h_1 s_{t-1} + \ldots + h_L s_{t-L}$$

where L is the memory of the filter.

- The coefficients $h_0, \ldots, h_L$ are called, the impulse response. The remain the same irrespective of t (thus, LTI filter).

The notion of filter

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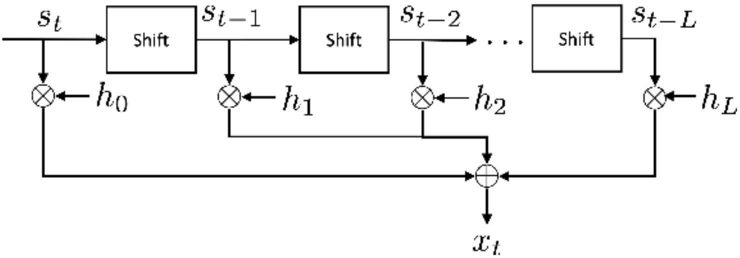

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Filtering in general

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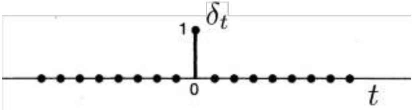
- In general a filter can combine an arbitrary number of samples in the sequence:
$$x_t = \sum_{\ell=-\infty}^{+\infty} h_\ell s_{t-\ell}$$
this operation is called *convolution* $x_t := h_t \star s_t$.
- Let us consider the impulse signal



- If the input is $s_t = \delta_t$, then the output is h_t (hence the name **impulse response** for h_t):

$$x_t = \sum_{\ell=-\infty}^{+\infty} h_k s_{t-\ell} = \sum_{\ell=-\infty}^{+\infty} h_k \delta_{t-\ell} \equiv h_t$$

Shift invariance

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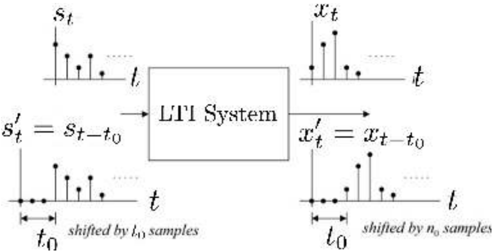
- Linear Time Invariant (LTI) refers to the fact thzat the **filter** is **a shift invariant operator**

Shift invariance

A linear system is a Linear Time Invariant (LTI) (or homogeneous) filter is shifting the input shifts the output by the same amount. Mathematically:

$$x_t = h_t \star s_t \quad \Leftrightarrow \quad x_{t-d} \equiv h_t \star s_{t-d}$$



Autoregressive and Moving average filters

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- A finite memory filter has the feedforward structure we showed before, and is called **Moving Average (MA)**:

$$x_t = \sum_{\ell=0}^{r} b_\ell s_{t-\ell}$$

- A filter that generate an infinite impulse response is teh **Autoregressive (AR)** model, which is a *feedback filter*:

$$x_t = \sum_{\ell=0}^{q} a_\ell x_{t-\ell} + b_0 s_t$$

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ARMA filter model

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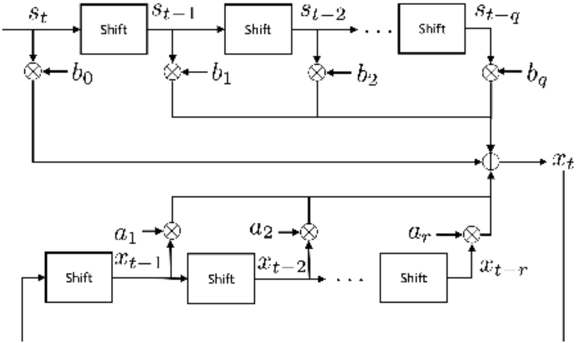
- The combination of the two is called **Auto-regressive Moving Average (ARMA)** filter:

$$x_t = \sum_{\ell=0}^{q} a_\ell x_{t-\ell} + \sum_{\ell=0}^{r} b_\ell s_{t-\ell}$$



ARMA models for random time series

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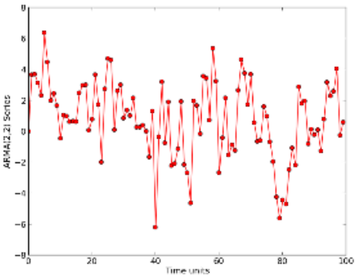
- What these filters abstractions are used for is primarily for modeling a time series

$$x_t = \sum_{\ell=0}^{q} a_\ell x_{t-\ell} + \sum_{\ell=0}^{r} b_\ell s_{t-\ell}$$

where $s_t \sim p(s_t)$ is assumed to be random i.i.d. in ARMA random processes.

- The idea is to say that there is an **excitation signal** s_t, that has independent samples, and that the observed signal x_t



Inference for ARMA

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- Note that ARMA series are Markov processes - x_t depends on the following past history $\boldsymbol{p}_t = [x_{t-1}, \ldots, x_{t-q}, s_{t-1}, \ldots, s_{t-r}]$

$$p(x_t|\text{all history prior to time } t) = p(x_t|\boldsymbol{p}_t)$$

- If the filter parameters $[a_1, \ldots, a_q[, [b_0, \ldots, b_r]$ are known, we can forecast the future samples:

$$\hat{x}_t = \overbrace{\sum_{\ell=0}^{q} a_\ell x_{t-\ell} + \sum_{\ell=1}^{r} b_\ell s_{t-\ell}}^{[a_1,\ldots,a_q,b_1,\ldots,b_r]\boldsymbol{p}_t^\top} + b_0 \mathbb{E}[s_t]$$

- With GSP filter models we want to perform similar tasks but across different nodes in the graph

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Time graph

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- The time index can be seen as a directed graph and the notion of neighborhood in time series is proximity in time ($v = t$, $\mathcal{N}_t = \{t-1, t+1\}$)

- Any *neighbohood* operation can involve only samples in $\mathcal{N}_t = \{t-1, t+1\}$

Time graph: linear neighborhood operations

- In general linear neighborhood operations have the form:

$$x_t = w_0 s_t + w_{-1} s_{t-1} + w_1 s_{t+1}$$

- A *time-shift* operation moves a signal forward in time (what was at time 0 is now at time 1 etc.) is a linear neighborhood operation

$$s_t \to x_t = s_{t-1}$$

  and it is an LTI filter with IR $h_t = \delta_{t-1}$.

- Another example is a local derivative:

$$\text{causal } x_t = s_t - s_{t-1}, \quad \text{non-causal } x_t = s_t - \frac{1}{2} s_{t-1} - \frac{1}{2} s_{t+1}$$

- Any filtering operation $h_t \star s_t$ can be interpreted as the composition of these elementary neighborhood operations

Outline

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3 Review of Digital Signal Processing
- The notion of filter
- Fourier Transform

Complex exponential inputs

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- When the input is an exponential $s_t \propto z^t$ (where $z \in \mathbb{C}$) the filters' shift invariant property is such that the output $x_t \propto z^t$
- Note that an exponential z^t is such that, if shifted, $z^{t-d} = z^t z^{-d}$. Let $s_t = z^t$ and calculate $x_t = h_t \star s_t = h_t \star z^t$

$$x_t = \sum_\ell h_\ell s_{t-\ell} = \sum_\ell h_\ell z^{t-\ell} = \sum_\ell h_\ell z^t z^{-\ell} = \overbrace{\left(\sum_\ell h_\ell z^{-\ell}\right)}^{\tilde{h}(z)} \overbrace{z^t}^{s_t}$$

- This proves when the input of an LTI filter is an exponential, the output is also an exponential proportional to the input

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Eigenvectors of LTI systems

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- Since, for an eigenvector input and output of a linear system are proportional another way of seeing this fact is to say that the signals $u_t = z^t$ are eigenvectors for LTI filters (a specific form of linear operators) because,

$$h_t \star z^t = \tilde{h}(z) z^t \quad \Leftrightarrow \quad h_t \star u_t \propto u_t$$

- The eigenvalues corresponding to $u_t = z^t$ are

$$\tilde{h}(z) = \sum_{\ell} h_\ell z^{-\ell}$$

which is also called the z-transform of h_t, or *transfer function* for the filter

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z-Transform and Fourier Analysis

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- The Discrete Time Fourier Transform (DTFT) is the z-transform evaluated at $z = e^{-j\omega}$ (i.e. the unit circle)

$$\tilde{h}(e^{j\omega}) = \sum_{\ell} h_\ell e^{-j\omega\ell}$$

- The z and Fourier transform are defined obviously not only for the filters but for the signals themselves:

$$\tilde{x}(z) = \sum_t x_t z^{-t}, \quad \tilde{s}(z) = \sum_t s_t z^{-t}$$

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Multiplication property

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- The other important relationship is as follows

$$x_t = h_t \star s_t \quad \Leftrightarrow \quad \tilde{x}(z) = \tilde{h}(z)\tilde{s}(z)$$

Proof:

$$\tilde{x}(z) = \sum_t x_t z^{-t} = \sum_t \Big(\sum_\ell h_\ell s_{t-\ell}\Big) z^{-t} = \sum_t \sum_\ell h_\ell s_{t-\ell} z^{-(t-\ell+\ell)}$$

$$= \sum_t \overbrace{\Big(\sum_\ell h_\ell z^{-\ell}\Big)}^{\tilde{h}(z)} s_{t-\ell} z^{-(t-\ell)}$$

$$= \tilde{h}(z) \sum_t s_{t-\ell} z^{-(t-\ell)} = \tilde{h}(z) \overbrace{\sum_{t'} s_{t'} z^{-t'}}^{\tilde{s}(z)}$$

Filters representation with the shift operator

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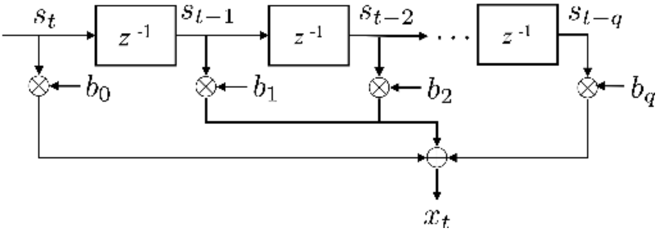

- Conventionally, in the filter block diagrams, we use z^{-1} to represent the time shift
- Shift invariance is obvious, since

$$\tilde{x}(z) = \tilde{h}(z)\tilde{x}(z) \quad \Leftrightarrow \quad \tilde{x}(z)z^{-d} = \tilde{h}(z)\tilde{s}(z)z^{-d}$$

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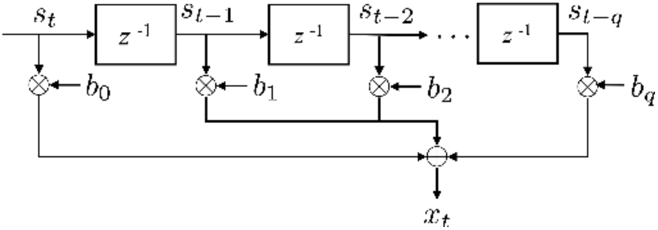

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Finite sequences

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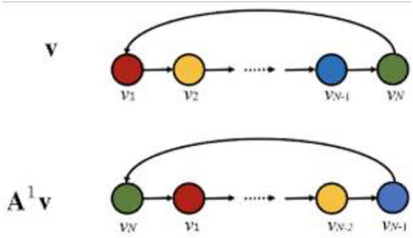
- Time series abstractions last forever $-\infty \leq t \leq +\infty$
- In reality we only have finite sequences for DSP. The same is true for graphs: they may be large but not infinite!
- If we have N values, we can assume that the rest of the samples are zero, i.e. $x_t = 0$ for $t < 0$ and $t > N-1$
- In DPS is algorithmically convenient to manipulate finite signals considering their *periodic extension* x_t^{PE} defined as follows

$$x_t^{\text{PE}} = \sum_{p=-\infty}^{+\infty} x_{t-pN}$$



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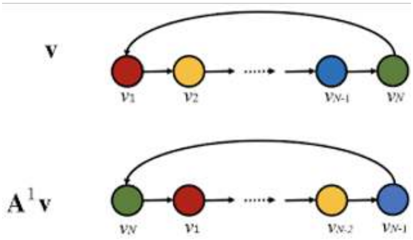

Periodic Extension

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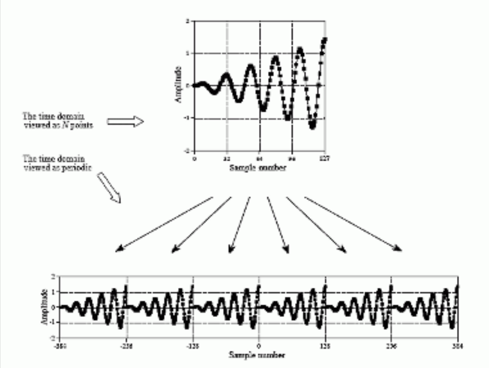

- The reason is that the Fourier Analysis of the signal can be has associated the discrete frequency components $u_t^{(n)} = e^{j2\pi \frac{nt}{N}}$

$$\tilde{x}_n = \frac{1}{N}\sum_{t=0}^{N-1} x_t e^{-j2\pi \frac{nt}{N}} \quad \Rightarrow \quad x_t^{\text{PE}} = \sum_{n=0}^{N-1} \tilde{x}_n e^{j2\pi \frac{nt}{N}}$$

Periodic Extension

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- The following operation is called the Discrete Time Fourier Transform (DFT) and it is equal to the DTFT evaluated at $\omega = 2\pi\frac{nt}{N}$, i.e.

$$\tilde{x}_n = \frac{1}{N}\sum_{t=0}^{N-1} x_t e^{-j2\pi\frac{nt}{N}}, \quad \Rightarrow \quad \tilde{x}(e^{j\omega t})\Big|_{\omega=2\pi\frac{nt}{N}} \equiv \tilde{x}_n$$

- The Periodic Extension is the Inverse Discrete Fourier Transform (IDFT):

$$x_t^{\mathsf{PE}} = \sum_{n=0}^{N-1} \tilde{x}_n e^{j2\pi\frac{nt}{N}} \qquad \Rightarrow \quad x_t^{\mathsf{PE}} \equiv x_t, \quad t = 0, \ldots, N-1$$

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Circular convolution

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- When the input of an LTI filter is periodic the output is also periodic. In fact, it can be shown that, if s_t is periodic

$$x_t = h_t \star s_t = \sum_{n=0}^{N-1} \overbrace{(\tilde{h}_n \tilde{s}_n)}^{\tilde{x}_n} e^{j2\pi \frac{nt}{N}}$$

$$\tilde{h}_n = \frac{1}{N} \sum_{t=0}^{N-1} h_t e^{-j2\pi \frac{nt}{N}}, \quad \tilde{x}_n = \frac{1}{N} \sum_{t=0}^{N-1} s_t e^{-j2\pi \frac{nt}{N}}$$

- However, if $x_t = h_t \star s_t$ has a general finite duration input s_t, if we convolve with h_t the periodic extension s_t^{PE} of s_t with h_t we do not get the periodic extension of the output x_t

$$h_t \star s_t^{\mathsf{PE}} \neq x_t^{\mathsf{PE}}$$

what we have is a circular convolution

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what we have is a circular convolution

Outline

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The Graph Shift Operator (GSO)

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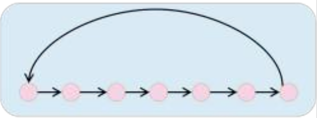
Point of view: algebraic SP, introduced in [1]

- For a (periodic/time-limited) signal the canonical coordinate vectors $\boldsymbol{x}$ form a cyclic graph G



$$\rightarrow \quad \boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

- The graph shift operator (GSO) $\mathbf{S}$ can only mix elements adjacent in G;
- A filter $\mathcal{H}(\mathbf{S})$ is a linear operator invariant relative to the GSO $\mathbf{S}$

$$\boldsymbol{x} = \mathcal{H}(\mathbf{S})\mathbf{s} \quad \rightarrow \quad \mathbf{S}\mathbf{x} = \mathbf{S}\mathcal{H}(\mathbf{S})\mathbf{s} \equiv \mathcal{H}(\mathbf{S})\mathbf{S}\mathbf{s} \tag{1}$$

Shift invariance

$$\mathbf{S}\mathcal{H}(\mathbf{S}) \equiv \mathcal{H}(\mathbf{S})\mathbf{S} \tag{2}$$

Graph Filters

Cornell University

- To be invariant w.r.t. the GSO a **graph filter** must be a matrix polynomial:

$$\mathcal{H}(\mathbf{S}) \triangleq \sum_{\ell=-\infty}^{+\infty} h_\ell \mathbf{S}^\ell. \tag{3}$$

- Graph/network data are **filtered** graph signals, i.e.,

$$\boldsymbol{x} = \mathcal{H}(\mathbf{S})\boldsymbol{s}. \tag{4}$$

- The signal/observation is $\boldsymbol{x}$ while $\boldsymbol{s}$ is viewed as the **excitation**.

Graph Filters Examples

Cornell University

- Examples of infinite order filters — $\mathcal{H}(\mathbf{S}) = (\mathbb{I} - \alpha\mathbf{S})^{-\mathbf{1}}$, $\mathcal{H}(\mathbf{S}) = \mathbf{e}^{\alpha\mathbf{S}}$.

Special cases for the GSO

The GSO in [2] is the adjacency operator $\mathbf{S} = \mathbf{A}$; in e.g.[3] the GSO is the Laplacian operator $\mathbf{S} = \mathrm{diag}(\mathbf{A}\mathbf{1}) - \mathbf{A}$, but there are other generalizations.

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Graph Laplacian

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- The Laplacian is popular choice for the GSO
  - A differential (not shift operator) - GSO is a misnomer used nonetheless
- Imperfect for models with a a directed graph support
- The Laplacian of a (weighted) graph is $\mathbf{S} = \boldsymbol{B}^T \mathrm{diag}(\boldsymbol{w}) \boldsymbol{B}$
- Note: the Laplacian has always a zero eigenvector equal to $\mathbf{1}$, since $\boldsymbol{B}^T \mathrm{diag}(\boldsymbol{w}) \boldsymbol{B}\mathbf{1} = 0$

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Graph Laplacian

Cornell University

- Using the Laplacian, with $0 \leq \gamma \leq \max_i [\mathbf{S}]_{ii}$ the operation

$$(\boldsymbol{x} - \gamma \mathbf{S} \boldsymbol{x}) \mapsto \boldsymbol{x} \qquad \text{or} \qquad \dot{\boldsymbol{x}} = -\mathbf{S}\boldsymbol{x}$$

  reallocates all the edge differences, adding them where an edge enters and subtracting them where an edge leaves the node

- A graph signal subject to this action is progressively *smoothened*

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What are low, versus high frequency?

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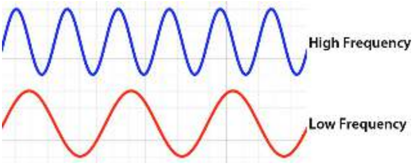
- The idea is to measure the *variation* in adjacent entries of the GFT basis: the larger the variation the higher the frequency.



- The total variation based on the squared Frobenius norm of graph derivative of the GS $\boldsymbol{x}$:

$$\mathrm{S}_2(\boldsymbol{x}) := \frac{1}{2}\|\nabla \boldsymbol{x}\|_{\mathrm{F}}^2 = \boldsymbol{x}^\top \mathbf{S}\boldsymbol{x} = \sum_{i,j} S_{ij}(x_i - x_j)^2. \qquad (5)$$

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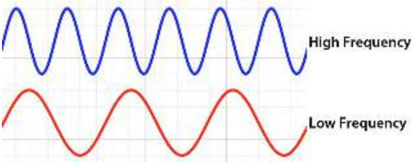

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GFT basis construction

Cornell University

- The GS is *smooth* if $\mathrm{S}_2(\boldsymbol{x})/\|\boldsymbol{x}\|_2$ is small.
- Note that the eigenvectors are minima of the ratio $\mathrm{S}_2(\boldsymbol{x})/\|\boldsymbol{x}\|_2$ which is called also Rayleigh quotient for the GSO $\mathbf{S}$:

$$\lambda_1 = \mathrm{S}_2(\boldsymbol{u}_1)/\|\boldsymbol{u}_1\|_2 = 0 = \min_{\boldsymbol{x}} \mathrm{S}_2(\boldsymbol{x})/\|\boldsymbol{x}\|_2$$

$$\lambda_i = \mathrm{S}_2(\boldsymbol{u}_i)/\|\boldsymbol{u}_i\|_2 = \min_{\boldsymbol{x}:\boldsymbol{u}_j\boldsymbol{x}^\top=0, j=1,\ldots,i-1} \mathrm{S}_2(\boldsymbol{x})/\|\boldsymbol{x}\|_2$$

Hence, the eigenvectors $\boldsymbol{U} = (\mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_n)$ with $\mathbf{u}_i \in \mathbb{R}^n$ being the eigenvector for λ_i in order $0 = \lambda_1 \leq \cdots \leq \lambda_N$ are the right basis and λ_i is a good parallel of the notion of frequency, going from low to high.

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Graph Fourier Transform (GFT)

- The Discrete-time Fourier Transform (DFT) basis are the eigen- vectors of the GSO for the cyclic graph $\rightarrow$ shift invariant signals
- $\mathbf{S} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$. The columns of $\boldsymbol{U}$ are the signals that are invariant w.r.t. the GSO $\mathbf{S}$, in fact $\lambda_i \boldsymbol{u}_i = \mathbf{S}\boldsymbol{u}_i$

$$\tilde{\boldsymbol{x}} = \overbrace{\boldsymbol{U}^{-1}}^{GFT} \boldsymbol{x} \tag{6}$$

- The transfer function of the GF is:

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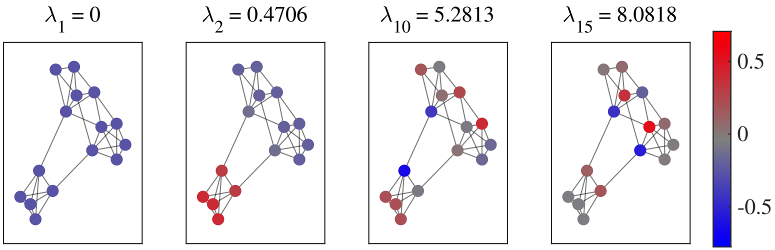
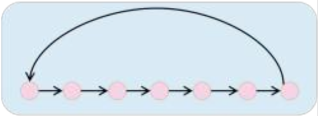
The GFT basis behavior



The GFT basis, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_{10}, \mathbf{u}_{15}$, associated to the graph Laplacian of an undirected, unweighted graph with 15 nodes

Consistency with DSP



$$\rightarrow \quad \boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

- This choice is consistent with the DFT for the cycle graph.
- In fact for the cycle graph the Laplacian is a circulant matrix:

$$\mathbf{S} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

and the eigenvectors and eigenvalues of $\mathbf{S}$, indexed by $k = 0, \dots, N-1$:

$$\boldsymbol{u}_k = (1, \rho_k, \dots, \rho_k^{N-1})^\top, \quad \rho_k = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi k}{N}}, \quad \lambda_k = 2(1 - \cos(2\pi k/N))$$

GFT filters input/output relatioship

Cornell University

- We said the transfer function of the GF $\mathcal{H}(\mathbf{S})$ is:

$$\tilde{\mathcal{H}}(\mathbf{\Lambda}) = \mathrm{diag}(\tilde{\boldsymbol{h}}) \quad \text{where} \quad [\tilde{\boldsymbol{h}}]_i = \tilde{h}_i = \sum_k h_k \lambda_i^k \tag{8}$$

- In fact, we recover the familiar relationship between the input and output of a filter GFTs, as the product with the filter transfer function:

$$\mathcal{H}(\mathbf{S}) = \sum_k h_k \mathbf{S}^k = \boldsymbol{U} \overbrace{\left(\sum_k h_k \mathbf{\Lambda}^k\right)}^{\tilde{\mathcal{H}}(\mathbf{\Lambda}) = \mathrm{Diag}(\tilde{\boldsymbol{h}})} \boldsymbol{U}^{-1}$$

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Outline

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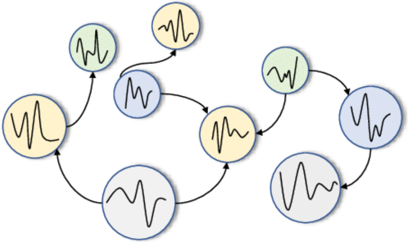
Graph Temporal Filters

Cornell University

- In many situations we observe time series of graph signals $\boldsymbol{x}_t$
- A graph-temporal filter with the impulse response:

$$\mathcal{H}(\mathbf{S}, t) := \sum_{\ell=-\infty}^{+\infty} h_{t,\ell}\mathbf{S}^{\ell},$$



- In the time domain:

$$\boldsymbol{x}_t = \sum_{k=-\infty}^{+\infty} \mathcal{H}(\mathbf{S}, k)\boldsymbol{s}_{t-k} \equiv \sum_{k=-\infty}^{+\infty} \sum_{\ell=-\infty}^{+\infty} h_{k,\ell}\mathbf{S}^{\ell}\boldsymbol{s}_{t-k}$$

Graph temporal filters

Cornell University

- The spatio-temporal GSO is $\mathbf{S}z^{-1}$. If we take the z-transform only:

$$\boldsymbol{x}_t = \sum_{k=-\infty}^{+\infty} \mathcal{H}(\mathbf{S};k)\boldsymbol{s}_{t-k}, \quad \mathcal{H}(\mathbf{S};k) = \sum_{\ell=0}^{L} h_{k,\ell}\mathbf{S}^{\ell}$$

$$\rightarrow \overbrace{\sum_t \boldsymbol{x}_t z^{-t}}^{\boldsymbol{x}(z)} = \sum_{k=-\infty}^{+\infty} \mathcal{H}(\mathbf{S};k) \overbrace{\sum_t \boldsymbol{s}_{t-k} z^{-t}}^{\boldsymbol{s}(z)z^{-k}} \rightarrow \boldsymbol{x}(z) = \overbrace{\left(\sum_{k=-\infty}^{+\infty} \mathcal{H}(\mathbf{S};k) z^{-k} \right)}^{\boldsymbol{h}(\mathbf{S};z)} \boldsymbol{s}(z)$$

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Finally, in the z-domain, the graph filter response is:

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Graph-z transform

Cornell University

- Having define the z-transform of the Graph Temporal filter, we now can define the joint Graph-z transform.
- Let $U_{\mathcal{V}}$ be the GFT basis of the spatial GSO $\mathbf{S} = U_{\mathcal{V}} \Lambda U_{\mathcal{V}}^{-1}$, of the graph temporal filters. We have:

$$\widetilde{x}(z) = U_{\mathcal{V}}^{-1} x(z) = U_{\mathcal{V}}^{-1} h(\mathbf{S}; z) s(z) = \sum_{\ell=-\infty}^{+\infty} h_{\ell}(z) U_{\mathcal{V}}^{-1} S^{\ell} s(z)$$

$$= \sum_{\ell=-\infty}^{+\infty} h_{\ell}(z) U_{\mathcal{V}}^{-1} \overbrace{(U_{\mathcal{V}} \Lambda U_{\mathcal{V}}^{-1})^{\ell}}^{=U_{\mathcal{V}} \Lambda^{\ell} U_{\mathcal{V}}^{-1}} s(z) = \overbrace{\sum_{\ell=-\infty}^{+\infty} h_{\ell}(z) \Lambda^{\ell}}^{\widetilde{h}(z)} \overbrace{U_{\mathcal{V}}^{-1} s(z)}^{\tilde{s}(z)}$$

$$= \widetilde{h}(z) \odot \widetilde{s}(z)$$

and $\tilde{h}(z)$ is the graph-temporal joint transfer function:

$$[\tilde{h}(z)]_i = \mathbb{H}(\lambda_i, z), \quad \text{where} \quad \mathbb{H}(\lambda, z) := \sum_{t=0}^{\infty} \sum_{k=0}^{K-1} h_{k,t} \lambda^k z^{-t}$$

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\begin{aligned}
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&= \sum_{\ell=-\infty}^{+\infty} h_\ell(z) \boldsymbol{U}_{\mathcal{V}}^{-1} \overbrace{(\boldsymbol{U}_{\mathcal{V}} \boldsymbol{\Lambda} \boldsymbol{U}_{\mathcal{V}}^{-1})^\ell}^{= \boldsymbol{U}_{\mathcal{V}} \boldsymbol{\Lambda}^\ell \boldsymbol{U}_{\mathcal{V}}^{-1}} \boldsymbol{s}(z) = \overbrace{\sum_{\ell=-\infty}^{+\infty} h_\ell(z) \boldsymbol{\Lambda}^\ell}^{\widetilde{\boldsymbol{h}}(z)} \overbrace{\boldsymbol{U}_{\mathcal{V}}^{-1} \boldsymbol{s}(z)}^{\tilde{\boldsymbol{s}}(z)} \\
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GF-ARMA (q,r) filters

Cornell University

- For graph signal processes a useful model is the GF-ARMA (q, r) filter
- Its input-output relation in time domain and z-GFT domain are:

$$\boldsymbol{x}_t - \mathcal{A}_1(\mathrm{S})\boldsymbol{x}_{t-1} \cdots - \mathcal{A}_q(\mathrm{S})\boldsymbol{x}_{t-q} = \mathcal{B}_0(\mathrm{S})\boldsymbol{s}_t + \cdots + \mathcal{B}_r(\mathrm{S})\boldsymbol{s}_{t-r},$$

$$\widetilde{\boldsymbol{a}}(z) \odot \widetilde{\boldsymbol{x}}(z) = \widetilde{\boldsymbol{b}}(z) \odot \widetilde{\boldsymbol{s}}(z),$$

Outline

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Graph signals with sparse GFT

Cornell University

- Like LTI filters, a graph filter can be classified as either low-pass, band-pass, or high-pass, or having a sparse support, depending on its graph frequency response
- Like in parametric spectrum estimation, low-pass graph signals can be viewed as the output of a low-pass graph filter with a excitation signal
- While generally speaking all such graph signals generate low-rank sequences of samples, are worth focusing on because:
  1. they are common in practice;
  2. the subspace vectors have interesting properties tied to the community structure of the graph

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Low Pass Graph Filter

Cornell University

- The definition of Ideal Low-Pass Graph Filter is rather obvious: for that $h(\lambda) = 1, \lambda \leq \lambda_k$ and 0 otherwise.
- Unfortunately, this is rather strict in practical models. Most GS models are amenable to the following looser definition:

Definition of Low-Pass Graph Signal

For any $1 \leq k \leq n-1$, define the ratio

$$\eta_k := \frac{\max\{|h(\lambda_{k+1})|, \ldots, |h(\lambda_n)|\}}{\min\{|h(\lambda_1)|, \ldots, |h(\lambda_k)|\}}. \tag{9}$$

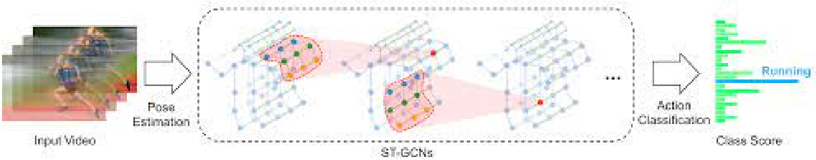
The graph filter $\mathcal{H}(\mathbf{S})$ is k-low-pass if and only if the low-pass ratio η_k satisfies $\eta_k \in [0, 1)$. Upon passing a graph signal through $\mathcal{H}(\mathbf{S})$, the high frequency components (above λ_k) are attenuated by a factor of less than or equal to η_k.

- The integer parameter k characterizes the *bandwidth*, or the cut-off frequency of the low-pass filter is at λ_k

Conclusions

Cornell University

- We introduced GSP
- We mentioned how frequently data exhibit a low pass nature that is due to the inherent structure of the network diffusion process
- Overall, for ML the key point is to be able to do feature extraction and for graph signals this requires identifying the right GSO



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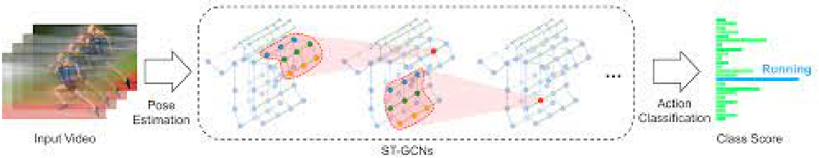

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