Lecture 16 - Financial Networks

Models for real networks

ECE 5260 – ORIE 5735

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Cornell Tech

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Content

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Review and overview

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- We introduced models for social dynamics
  - Models capturing herding
  - Models capturing polarization
  - Models where agents directly respond to actions
- Today we study financial networks
- We also briefly discuss collaboration networks

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Outline

1 Review

2 Financial networks

3 Collaboration Networks

4 Conclusions

Outline

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1 Review
- Models for social dynamics
- Voting models

DeGroot Opinion Dynamics

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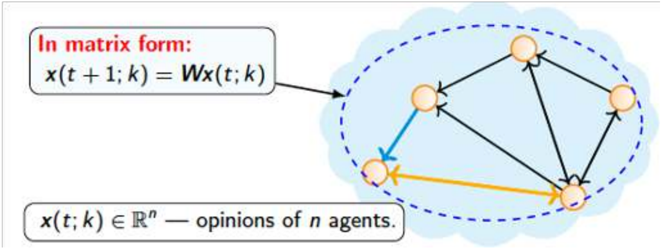
- DeGroot model: Agents takes a weighted average of neighbors' opinions, like in the average consensus dynamics.



- In the randomized version $\boldsymbol{W}(t)$ at each t is random.

if G is ***strongly connected*** *and* $\boldsymbol{W}\mathbf{1} = \mathbf{1}$, $\lambda_1(\boldsymbol{W}) = 1$, $\lambda_2(\boldsymbol{W}) < 1$

$$\lim_{t \to \infty} \boldsymbol{x}(t;k) = c(k)^\star \mathbf{1} \Longrightarrow \textit{consensus is reached.}$$

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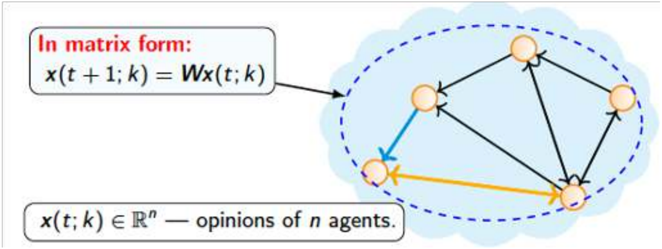

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DeGroot Model with Stubborn Agents

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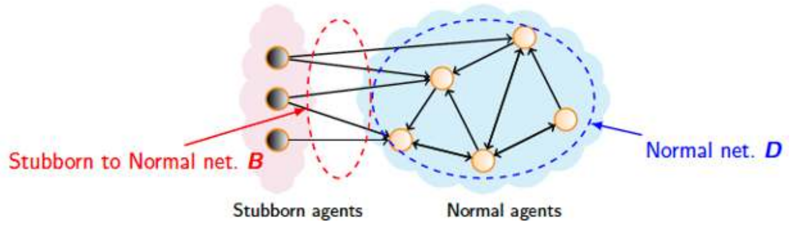
$$\boldsymbol{z}(t+1;k) = \boldsymbol{z}(t;k)$$
$$\boldsymbol{y}(t+1;k) = \boldsymbol{D}\boldsymbol{y}(t;k) + \boldsymbol{B}\boldsymbol{z}(t;k)$$ is an affine mapping!



Steady-state: *if sub-graph $\mathcal{V} \setminus \mathcal{S}]$ is* ***strongly connected*** *(+other conditions we discussed for affine dynamics):*

$$\boldsymbol{y}^k := \lim_{t\to\infty} \boldsymbol{y}(t;k) = (\boldsymbol{I} - \boldsymbol{D})^{-1}\boldsymbol{B}\boldsymbol{z}^k.$$

Bounded Confidence and Polarization

- R. Hegselmann and U. Krause: $d_{ij}(t;k)$ (e.g. $d_{ij}(t;k) = |x_i(t;k) - x_j(t;k)|$) is the opinion distance of agents i and j, agents "talk" iff $d_{ij}(t;k) < \tau$

$$x_i(t+1;k) = \sum_{j=1}^{n} W_{ij} u(\tau - d_{ij}(t;k)) x_j(t;k)$$

$$\boldsymbol{x}(t+1;k) = \boldsymbol{W}(\boldsymbol{x}(t;k))\boldsymbol{x}(t;k)$$

Phase transition to consensus

Let $\overline{W}_{ij}$ probability of the interaction and $d_{ij}(t;k) = |x_i(t;k) - x_j(t;k)|$. A **necessary** condition for the system to converge almost surely to consensus:

$$\tau > \overline{d}_k := \sum_{ij} \overline{W}_{ij} d_{ij}(0;k)$$

Trends

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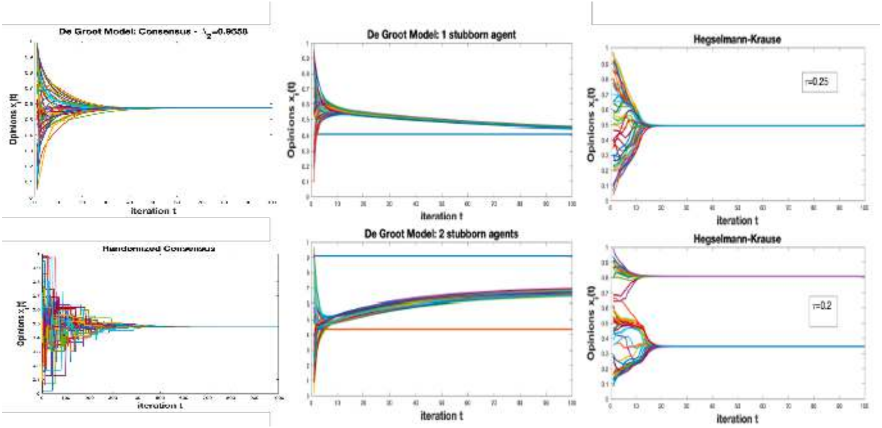

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Bounded opinions: Voter model

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- Agents have a prior belief $x_i(0;k) = p(\theta_k|s_i)$ on a topic θ_k
- Agents can only observe actions $a_i(t;k)$ (typically binary, i.e. $\in \{1,0\}$) and update their beliefs

Variants of the voter model

- Rational agents with bounded opinions - Bikhchandani, Hirshleifer and Welch (BHW)
- Binary opinion dynamics - Peter Clifford and Aidan Sudbury '73 is inspired by Ising model for spin alignment

Both of them lead to consensus or herding

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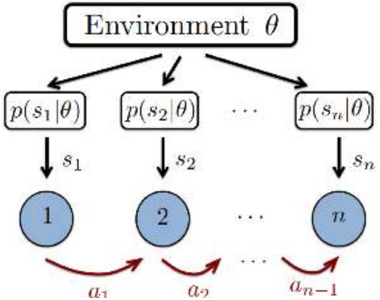
Herding of rational agents

- Bikhchandani, Hirshleifer and Welch (BHW): θ is randomly determined between the values of 1 ("good state") and 0 ("bad state")
- The agents gain c if $a_i = \theta = 1$ they loose c if $a_i = 1$ and $\theta = 0$ and $a_i = 0$ yields no gain or cost



The updates are in sequence. The tth agent to update observed s_t and the actions of the previous agents to decides

$$a(t;k) = \text{argmax}_{\theta \in \{0,1\}} P(\theta | s_t, a(t-1;k), \ldots, a(1;k)) \rightarrow \text{rational choice}$$

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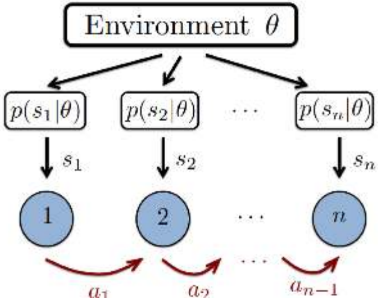

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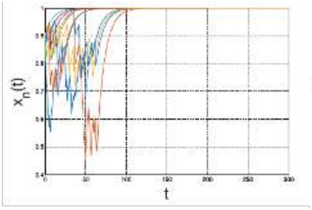
Voter model - Clifford and Sudbury

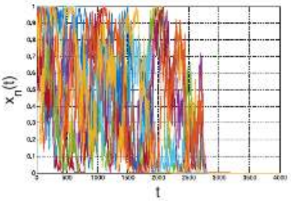
- In interacting social agents copy at random the action of a neighbor. $a_i(t;k)$ is equivalently a Bernoulli random variable:

$$a_i(t;k) \sim \mathcal{B}(x_i(t;k)), \qquad x_i(t+1;k) = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i}^{n} a_j(t;k)$$

All actions are almost surely the same if the network is finite and connected, i.e. $t \to \infty$ $\boldsymbol{a}(t;k) = a^{\infty}$ a.s.. The final collective action can be $a^{\infty} = 0$ or $a^{\infty} = 1$. Given $x_i(0;k)$:

$$P(a^{\infty} = 1) = \frac{1}{n} \mathbf{1}\mathbf{1}^T \boldsymbol{x}(0;k).$$





Voter model with stubborn agents

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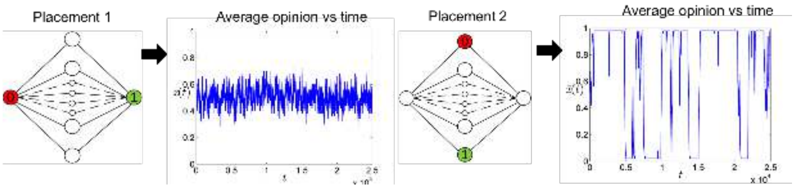
- No consensus. The network continues to randomly change votes

Analysis

If every non-stubborn node in the network has at least one directed path to a stubborn agent votes $\boldsymbol{a}(t;k)$ converges in distribution, i.e. for $t \to +\infty$ $\boldsymbol{a}(t;k) \sim \mathcal{B}(\boldsymbol{x}(+\infty))$.



Outline

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Connection between opinion and financial

- Up to now we used networks to model how beliefs and actions spread
- Today we show that very similar mathematics models how value, risk, and demand propagate.
- Different domains, but very similar network mathematics: averaging for opinions with stubborn agents resemble the affine balance equations for value and production, and nonlinear thresholds for cascades.

Modeling the value of firms

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- There are several models that closely resemble the social network dynamics discussed before. Next, I will make an example.
- Suppose there the set of firms is $\mathcal{V}$, with $|\mathcal{V}| = N$.
- These firms have a fraction of M **primitive assets** in a set $\mathcal{M}$
- A simple model for the firm value x_i at equilibrium is as follows:

$$x_i = \sum_{j \in \mathcal{V}} W_{ij} x_j + \sum_{j = \mathcal{M}} B_{ij} p_j$$

where $W_{ij} x_j$ is the fraction of the value belonging to other firms that are shareholders, $B_{ij} p_j$ is the fraction of the **value** p_j of the jth primitive asset owned by firm i.

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Modeling the value of firms

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- Being fractions of a unit value, we have that $\sum_j W_{ij} + \sum_j B_{ij} = 1$, which means that $\sum_j W_{ij} < 1$

Assets Value Propagation Master Equation

The firms network edge set corresponds to the $W_{ij} > 0$. The weight matrix $\boldsymbol{W}$ is substochastic $\boldsymbol{W}\mathbf{1} < \mathbf{1}$ We can use the two matrices $\boldsymbol{W}$ and $\boldsymbol{B}$ of these weights and write

$$\boldsymbol{x} = \boldsymbol{W}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{p} \quad \Rightarrow \quad \boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{W})^{-1}\boldsymbol{B}\boldsymbol{p}$$

where note that this is the same solution of an affine system

Valuation of the firm

v_i is a fraction $c_i x_i$ of its *in the book* value, i.e.:

$$\boldsymbol{v} = \mathrm{diag}(\boldsymbol{c})\boldsymbol{x} = \mathrm{diag}(\boldsymbol{c})(\boldsymbol{I} - \boldsymbol{W})^{-1}\boldsymbol{B}\boldsymbol{p}$$

Firm economic shocks

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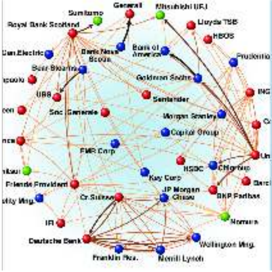
- The model discussed above has been used, for example, to study how failures propagate across a network of firms
- The hypothesis is that if the valuation of the firm drops below a threshold, $v_i < \overline{v}_i$, then the firm experiences a loss ℓ_i.



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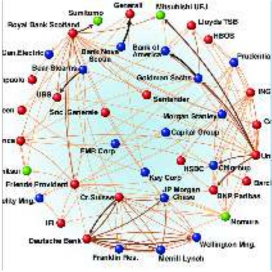

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- The vector of losses be written as $\mathrm{diag}(\boldsymbol{\ell})\mathbb{1}_{\boldsymbol{v}<\overline{\boldsymbol{v}}}$, where $\mathbb{1}_{\boldsymbol{x}}$ is an indicator vector such that $[\mathbb{1}_{\boldsymbol{x}}]_i = 1$ if $x_i < 0$ and $[\mathbb{1}_{\boldsymbol{x}}]_i = 0$ else
- The model is non-linear since now:

$$\boldsymbol{x} = \boldsymbol{W}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{p} - \mathrm{diag}(\boldsymbol{\ell})\mathbb{1}_{\boldsymbol{v}<\overline{\boldsymbol{v}}},$$

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- Hence, the amount of loss that can propagates from firm j to firm i is $[\text{diag}(\boldsymbol{c})(\boldsymbol{I} - \boldsymbol{W})^{-1}]_{ij}\ell_j$
- The analysis of this system is relatively complex.
- What is of interest is to understand what is the impact of the parameters of the model $\Psi = (\boldsymbol{c}, \boldsymbol{W}, \boldsymbol{B}, \boldsymbol{p}, \boldsymbol{\ell})$

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Firm economic shocks

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Failure propagation

Let the set of parameters $\boldsymbol{\Psi}'$ relative to $\boldsymbol{\Psi}$ be such that $\boldsymbol{p}' = \boldsymbol{p}$ but $[\text{diag}(\boldsymbol{c}')(\boldsymbol{I} - \boldsymbol{W}')^{-1}]_{ij} > \text{diag}(\boldsymbol{c})(\boldsymbol{I} - \boldsymbol{W})^{-1}]_{ij}$, $i \neq j$. If a cascade affects the system with parameters $\boldsymbol{\Psi}$, then it will affect that with $\boldsymbol{\Psi}'$

- The condition is true of organizations hold more of each other's investments $\Rightarrow$ we face more failures in any given cascade.
- **Insight**: Integration is only good if we can avoid first failures.
- There is a trade-off: integrating can eliminate some first failures. However, given that a first failure occurs, integration only exacerbates the resulting cascade.

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2 Financial networks

- Leontief analysis of inter-relationship between economic sectors

Inter- relationship between economic sectors

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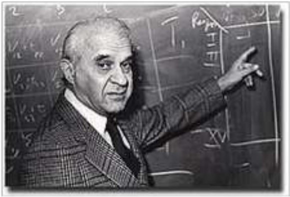
Table 2

Purchases by / Sales of	Industry 1	Industry 2	Industry n	Total sales
Industry 1	x_{11}	x_{12}	x_{1n}	$X_1 = \sum_j x_{1j}$
Industry 2	x_{21}	x_{22}	x_{2n}	$X_2 = \sum_j x_{2j}$
...	...	...	...	...
Industry n	x_{n1}	x_{n2}	x_{nn}	$X_n = \sum_j x_{nj}$
Total purchases	X_1	X_2	X_n	



- W. Leontief was on of the first to model and analyze the inter-relationship between economic sectors
- The idea was to leverage I/O tables, summarizing how the products (**outputs**) of a given industry or economic sector are used as **input** to other industries or sectors within the same, or different, economies[1]. Check https://www.bls.gov/emp/data/input-output-matrix.htmthe data from the Bureau of Labor Statistics.

[1] In the case of Import/Export exchanges with other countries.

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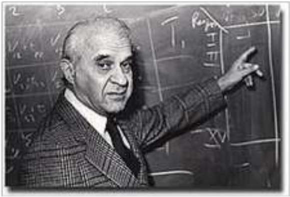

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Example

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The input-output table for two sectors (e.g., Agriculture and Manufacturing) is given below:

	Agriculture	Manufacturing	Final Demand
Agriculture	$x_{AA} = 30$	$x_{AM} = 20$	$b_A = 50$
Manufacturing	$x_{MA} = 10$	$x_{MM} = 40$	$b_M = 70$
Total Output	$o_A = 100$	$o_M = 120$	

Table: Leontief Input-Output Table for Two Sectors

The Leontief matrix $\boldsymbol{W}$ is the non negative matrix defined as:

$$\boldsymbol{W} = \begin{bmatrix} W_{AA} & W_{AM} \\ W_{MA} & W_{MM} \end{bmatrix} = \begin{bmatrix} \frac{x_{AA}}{o_A} & \frac{x_{AM}}{o_M} \\ \frac{x_{MA}}{o_A} & \frac{x_{MM}}{o_M} \end{bmatrix} = \begin{bmatrix} \frac{30}{100} & \frac{20}{120} \\ \frac{10}{100} & \frac{40}{120} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.167 \\ 0.1 & 0.333 \end{bmatrix}$$

Observations on the Leontief matrix

- The weight matrix $\boldsymbol{W}$ is constructed by dividing each entry by the total output of the sector.
- W_{ij}, for $i \neq j$, is the percentage of the total units of product i necessary to produce one unit of product j
- $W_{ij} \geq 0$, and the sum of each column is $< 1 \quad \Rightarrow \boldsymbol{W}$ is column sub-stochastic

Meeting the demand

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- The external demands of products from the two sectors are b_A and b_M respectively. How much must be produced by each sector to meet the demand?
- Naively, one would say, produce b_A and b_M
- The answers ignores that some of the output of each sector is not directly used to meet the demand, but is consumed by other sectors (even itself) to function, which is what the table is representing.
- This fact makes the matrix sub-stochastic...

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Meeting the demand

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Leontief's inverse

Cornell University

- To meet external demand $\boldsymbol{b} = (b_1, b_2)^\top$ the *economy* must produce $\boldsymbol{x} = (x_1, x_2)^\top$ units of product so that they satisfy

$$\boldsymbol{x} = \boldsymbol{W}\boldsymbol{x} + \boldsymbol{b}$$

- $\boldsymbol{W}\boldsymbol{x}$ is the so-called **intermediate demand**, products, created and consumed in the production process
- We recognize the affine equations, and their solution is:

$$\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{W})^{-1}\boldsymbol{b}$$

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Interpretation of the Leontief Inverse

In the example before:

$$\boldsymbol{I} - \boldsymbol{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.167 \\ 0.1 & 0.333 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.167 \\ -0.1 & 0.667 \end{bmatrix}$$

$$(I - \boldsymbol{W})^{-1} = \begin{bmatrix} 1.48 & 0.37 \\ 0.22 & 1.56 \end{bmatrix}$$

Each element of $(I - \boldsymbol{W})^{-1}$ is the *total (direct + indirect) increase in output* needed in sector i to meet an additional unit of final demand in sector j, e.g.:

- $(1, 1) = 1.48$: To meet one extra unit of final demand in *Agriculture*, total Agriculture output must increase by *1.48 units*.
- $(1, 2) = 0.37$: To meet one extra unit of final demand in *Manufacturing*, total Agriculture output must increase by *0.37 units*.
- $(2, 1) = 0.22$: To meet one extra unit of final demand in *Agriculture*, total Manufacturing output must increase by *0.22 units*.
- $(2, 2) = 1.56$: To meet one extra unit of final demand in *Manufacturing*, total Manufacturing output must increase by *1.56 units*.

I/O relationship money flow table

Cornell University

- The approach can be applied reverse flow of money from a sector j that buys a certain amount X_{ji} of units produced by sector i

- In this case let M_{ij} be the flow of money from sector j to sector i, and assume that M_{i0} is the money that comes from external demand b_i.
- The money sector i receives from sector j is

$$M_{ij} = X_{ji}\rho_i$$

where ρ_i is the price of sector's i product.

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- Like before, this point of view helps defining steady state equations for the value of the gross output produced by a sector
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- We define the matrix whose entries are the fraction of money node j pays to node i over its total payements to other firms:

$$[\boldsymbol{D}]_{ij} = D_{ij} = \frac{M_{ij}}{\sum_{k=0}^{N} M_{kj}}$$

and note that $\sum_{j=1}^{N} [\boldsymbol{D}]_{ij} < 1$ (is column sub-stochastic)

- Let the **gross output of sector** i be $[\boldsymbol{y}]_i$. Given the contribution from the external demand at equilibrium is:

$$\boldsymbol{y} = \boldsymbol{D}\boldsymbol{y} + \boldsymbol{f}, \quad \boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{D})^{-1}\boldsymbol{f}$$

where $[\boldsymbol{f}]_i$ is the $i$the sector output's value that flows to the sector from the final/external demand $b_i \rho_i$

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Upstreamness

Cornell University

- From this equation economists defined measures what they call upstreamness in a financial network, to understand global value chains (GVC)
- These metrics are aimed at defining the **relative position of a sector** in the value production chain.
- Industries selling a disproportionate share of their output to relatively upstream industries should be relatively upstream themselves

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Upstreamness

Cornell University

- The following metric was first proposed by Fally (2012)

$$u_i = 1 + \sum_j \frac{D_{ij} y_j}{y_i}$$

  where the first term is essentially $y_i / y_i = 1$.
- Large u_i means that the sector is receiving a lot of money from other sectors within the local financial network
- We can calculate $\boldsymbol{u}$ numerically knowing $\boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{D})^{-1} \boldsymbol{f}$

Upstreamness

Cornell University

- Let us define the matrix:

$$[\boldsymbol{A}]_{ij} = A_{ij} = \frac{M_{ij}}{\sum_{k=0}^{N} M_{ik}}$$

- Note that $[\boldsymbol{A}]_{ij}$ and $[\boldsymbol{D}]_{ij}$ have the same numerator M_{ij} but the normalization is different:
  - in $[\boldsymbol{A}]_{ij}$ the money flow is by the total money that i receives, i.e. $\sum_{k=0}^{N} M_{ik}$
  - in $[\boldsymbol{D}]_{ij}$ is normalized by the total money that i spends $\sum_{k=0}^{N} M_{kj}$

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Upstreamness at steady state

Cornell University

- When the global value chains become efficient the money a sector receives for its goods should be the same that are spent

$$y_i = \sum_{k=0}^{N} M_{ik} = \sum_{k=0}^{N} M_{ki}$$

- Hence, at steady state the upstreamness expression can be simplified as follows:

$$u_i = 1 + \sum_{j=1}^{N} \frac{D_{ij} y_j}{y_i} = 1 + \sum_{j=1}^{N} \frac{M_{ij}}{\cancel{\sum_{k=0}^{N} M_{kj}}} \frac{\cancel{\sum_{j=0}^{N} M_{kj}}}{\sum_{k=0}^{N} M_{ik}}$$

$$= 1 + \sum_{j=1}^{N} \frac{M_{ij}}{\sum_{k=0}^{N} M_{ik}} = 1 + \sum_{j=1}^{N} A_{ij}$$

- In matrix form:

$$\boldsymbol{u} = \boldsymbol{1} + \boldsymbol{A}\boldsymbol{1} \quad \Rightarrow \quad \boldsymbol{u} = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{1}$$

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Outline

Relational Networks

- In social networks influence is latent
- In financial network is that some or real network links may not public public
- Collaboration networks come from affiliation networks and are easier to build because hard data exist, like coauthorships

Collaboration networks

Cornell University

- The models for relational networks include so called **collaboration networks** as well.
- These networks can be build based on affiliation networks. The network bipartite graph linking a set of individuals to a set of activities that they have contributed to.
- In general the collaboration network graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the projection of the bipartite graph linking individuals that perform an activity together.
- These are classes of social networks but their advantage is that it is easier to find hard data about affiliation

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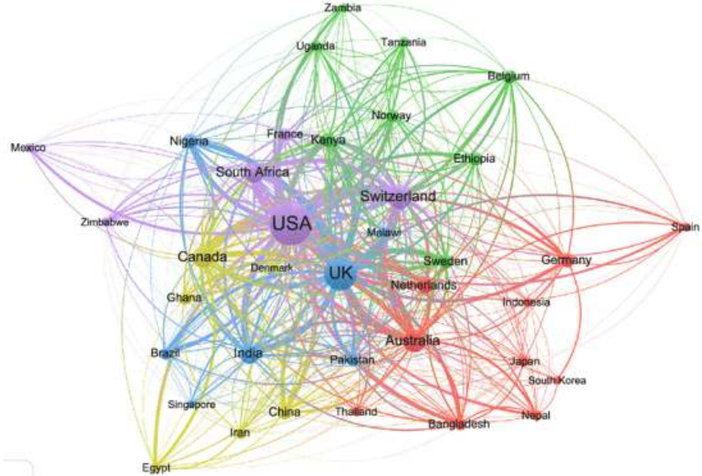
Scientific collaboration networks

Cornell University

- One of such cases is that of scientific networks: we know what are the coauthors in published papers.



Scientific collaboration networks

- In this case the network graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the projection of the bipartite graph linking authors and papers, i.e.:
  1. nodes $\mathcal{V}$ that are all the researchers that have published in a certain litterarature corpus in a certain field
  2. edges $\mathcal{E}$ that connect two researchers if they have coauthored a publication

Scientific collaboration networks studies

Cornell University

- In this case most studies have not captured *dynamics*, but rather focused on the graph itself and its features
- The type of analysis that has been done is looking at the metrics we studied in the previous module:
  - Degree distribution of the bipartite graph (i.e. number of papers and number of authors per paper)
  - Degree distribution of the collaboration network
  - Centrality measures
  - Clustering coefficient
  - Discovery of communities through graph clustering
- More recent work has focused on link prediction

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Scientific collaboration networks studies

Cornell University

A number of conclusions were drawn that apply to scientific collaboration networks in most research areas

1. The number of papers and the degree distribution p_k, exhibit fat tail that eventually is cut-off by an exponential trend. For both, with different parameters

$$p_k = Ck^{-\gamma}e^{-k/\beta}$$

2. The clustering coefficient is on average large and locally depends on the vertex degree, signaling *structure* in the network.
3. The degrees of the nearest-neighbor vertices are positively correlated, i.e. the network is **assortative**.
4. 80-90% of the nodes are connected (i.e. there exist a giant component)

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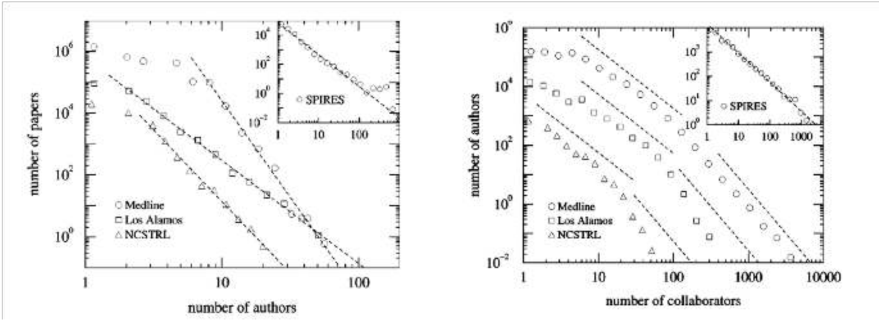
Analysis of scientific collaboration networks

Cornell University

- Examples of the number of papers per authors and degree distribution are shown in the figures below



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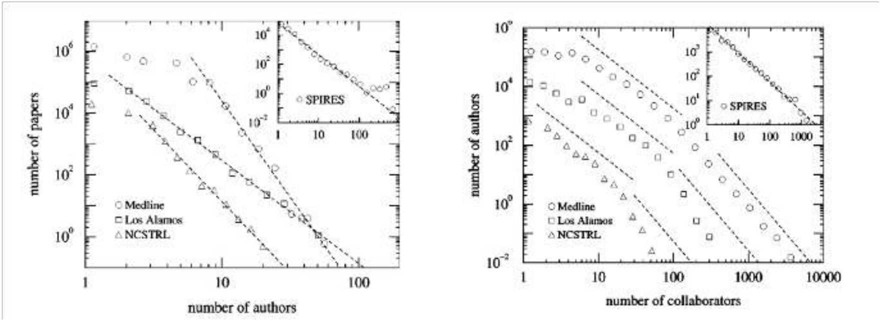

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Outline

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1 Review

2 Financial networks

3 Collaboration Networks

4 Conclusions

Concluding remarks

Cornell University

- In this lecture we concluded our discussion on relational networks
  - We discussed financial networks and financial shocks and saw how the Leontief matrix inverse is ubiquitous
  - We discussed how to use data to construct collaboration networks
- These models were example of linear, non-linear and affine dynamics that we introduced in general before

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