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Lecture 13 - General Dynamics, Flow Networks

Examples of network dynamics
ECE 5260 – ORIE 5735

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Review and overview

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- The goal of this module is to understand how to model dynamics over networks
- Last time we completed the discussion about the consensus protocol considering the case of random local interactions
- We considered a first case of network interaction that is non-linear: epidemic models
- We looked at general non-linear models
- Today we are going to define flow network dynamics, which are a special case

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Outline

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1. Review
   - Review of epidemic models
   - New: Agent Based Models for epidemic studies
   - Review of general analysis of network dynamics

Example 1: SI model

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Susceptible-Infected (SI) Epidemic Model

The model is on $x_i(t) =$ the probability agent i is infected. With β is the probability of contact with a neighbor, the ODE are

SI model $$\dot{x}_i = \sum_{j\in\mathcal{N}_i}(1-x_i)x_j \equiv \beta(1-x_i)\sum_{j\in\mathcal{N}_i} x_j, \quad \forall i \in \mathcal{V}$$

The mean field equation, is the so called Logistic ODE:

$$\dot{x}^{\text{ave}} = \beta(1-x^{\text{ave}})x^{\text{ave}}$$

and has solution:

$$x^{\text{ave}}(t) = \frac{x^{\text{ave}}(0)e^{\beta t}}{1-x^{\text{ave}}(0)+x^{\text{ave}}(0)e^{\beta t}}$$

Logistic differential equation trend

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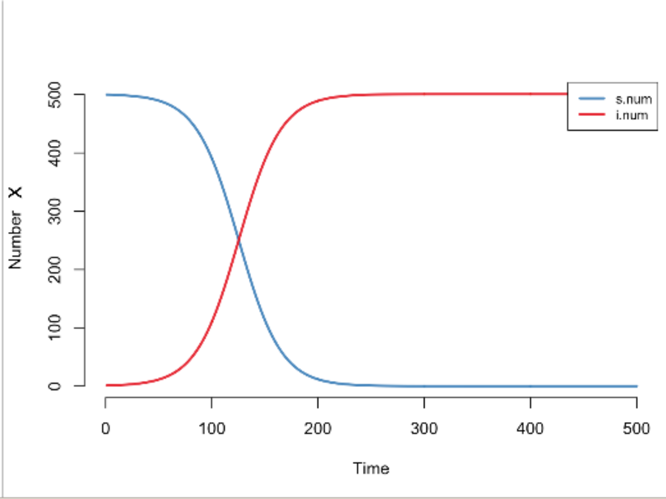
- The trend of the curve is a sigmoid. The rate at which the susceptible agents decrease is $s^{\text{ave}}(t) = 1 - x^{\text{ave}}(t)$



Network Dynamics in the SI model

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- Are analyzed at the epidemic onset, when $x_i(t) \approx 0$

$$\dot{x}_i = \beta(1 - x_i) \sum_{j \in \mathcal{N}_i} x_j \approx \beta \sum_{j \in \mathcal{N}_i} x_j, \quad \Rightarrow \quad \dot{\boldsymbol{x}} \approx \beta \boldsymbol{A} \boldsymbol{x}$$

- The solution is

$$\boldsymbol{x}(t) \approx \sum_{k=1}^{N} \xi_k(0) e^{\beta \lambda_k t} \boldsymbol{u}_k, \quad t \text{ small}$$

where $\boldsymbol{u}_k$ are the eigenvectors and eigevalues of the adjacecy matrix $\boldsymbol{A}$ and $\boldsymbol{\xi}(0) = \boldsymbol{U}^{-1} \boldsymbol{x}(0)$

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Example 2: SIR model

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Susceptible Infected Removed (SIR) model

Let γ the probability that a node infected recovers. The probability that the node is removed $r_i(t) = \gamma x_i$. The dynamics of the infection probability for node i $x_i(t)$ are:

SIR model $$\dot{x}_i = \sum_{j \in \mathcal{N}_i} \beta(1 - x_i)x_j - \gamma x_i,$$

- The analysis at the epidemic onset onset, when $x_i(t) \approx 0$ yields

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} \beta(1-x_i)x_j - \gamma x_i \approx \sum_{j \in \mathcal{N}_i} \beta x_j - \gamma x_i, \ \Rightarrow \dot{\boldsymbol{x}} \approx (\beta \boldsymbol{A} - \gamma \boldsymbol{I})\boldsymbol{x}$$

- The solution depends on the EVD of $(\beta \boldsymbol{A} - \gamma \boldsymbol{I})$

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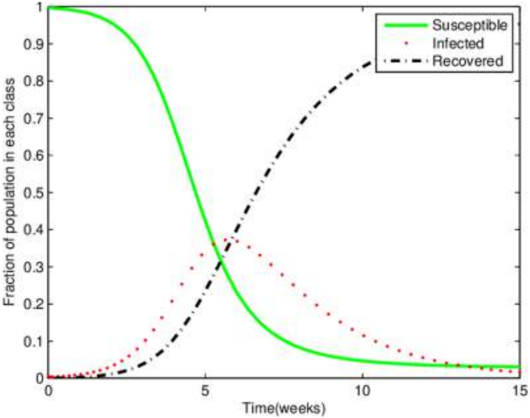
Solving the SIR model numerically

- In general the solution is calculated relatively easily numerically and looks as follows



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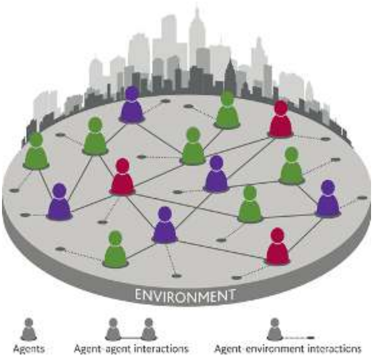
Simulation tools: Agent-Based Models (ABMs)

- The SIR model is considered the golden standard in terms of getting insights from data.
- When simulating the dynamcs one can model the social contact of individuals.
- The SIR model is a special case of what computational epidemiology refers to as Agent-Based Models (ABMs)



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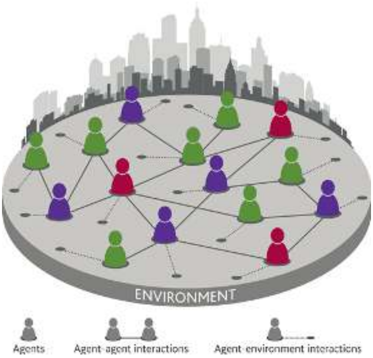

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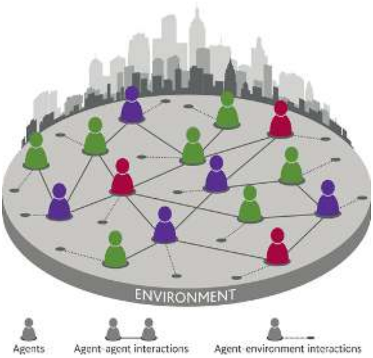

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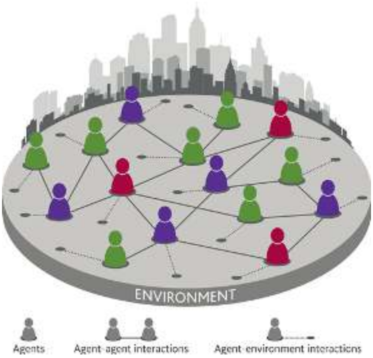

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Why ABMs?

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- In computational epidemiology, ABMs have been used and comparatively evaluated against SIR models to:
    1. accurately reflecting known dynamics and real data
    2. simulating data-driven stochastic heterogeneity across agent populations
    3. integrating economic considerations
    4. allowing detailed simulation of known public policy measures at different times, intensities and dates, and
    5. providing a simple interface for non-expert users to configure and interpret.
- Analysis of the behavior of both classes of models suggests the two classes will give qualitatively the same result.
- ABMs have an advantage in accounting for heterogeneity in subpopulations.

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ABMs modeling components

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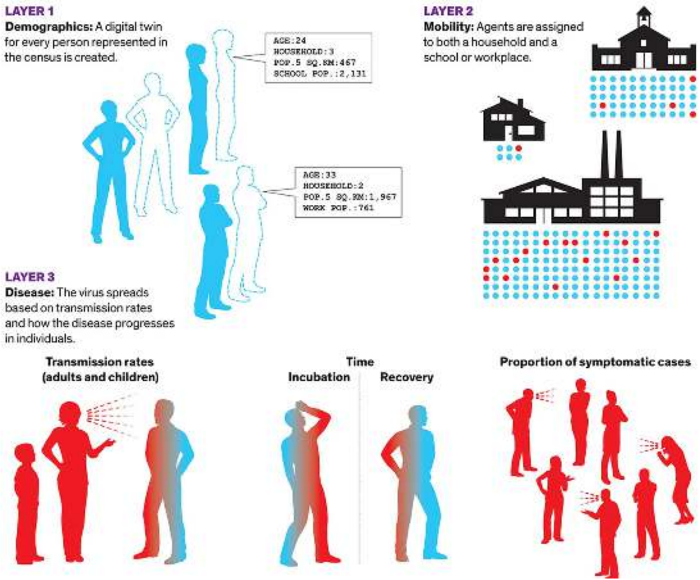

ABMs additional features

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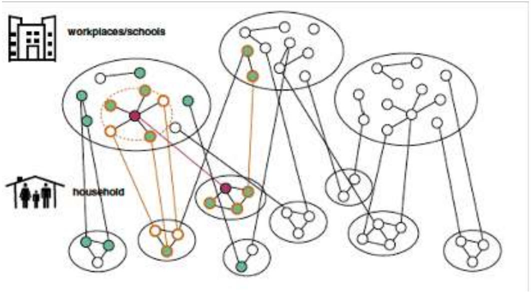
- Spatial motion of agents can be simulated as a random walk.
- This includes dwelling in a specific place (self-loop) that causes the encounter (in SIR this is simply $\mathcal{E}$ and the rate of meeting β)



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- Population density can also be simulated by shriking the size of the area.

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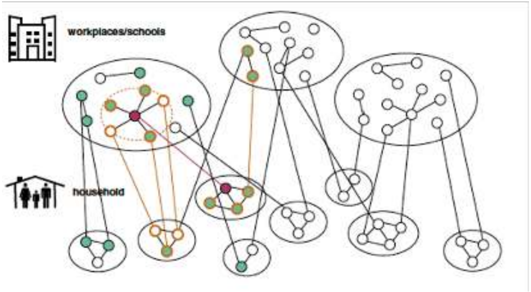

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ABMs features

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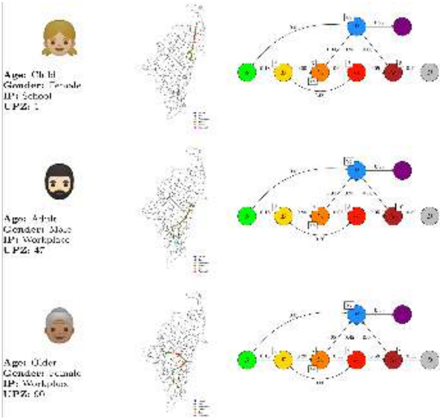
- **Demographics**: Each node can be labeled with age, sex, co-morbidities that change the type of infection outcomes.
- **Disease**: Agents can be in several states: Susceptible and Recoverd, but also Infected Symptomatic (Severe and not Severe), Infected Asymtomatic (Detected and Undetected), and Dead. They can have different transmission rates.



- An example of ABM code for COVID-19 studies is in https://www.medrxiv.org/node/92210.external-links.html

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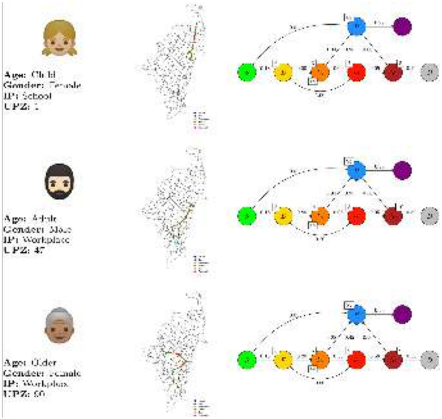

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Agent based simulation software

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- The simulations described are tailored to epidemics, but there are a number of other cases that are amenable to agent based modeling.
- An example of an agent based simulation software is called **Mesa** https://github.com/topics/agent-based-simulation
  - It is an open-source Python library for simulating complex systems and exploring emergent behaviors (from forest fire, to social polarization, to bank reserves...).
  - It can be a useful resource to simulate data for a final project.

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General analysis

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- We said that we want to looking locally at the trends around a fixed point $\boldsymbol{x}^*$ where $\dot{x}_i(t) = 0$ for all $i \in \mathcal{V}$

$$f(x_i^*) + \sum_{j \in \mathcal{N}_i} g(x_i^*, x_j^*) = 0, \quad \forall i \in \mathcal{V}$$

  perturbing the state by a small amount $\boldsymbol{\epsilon}$, i.e. $\boldsymbol{x}(0) = \boldsymbol{x}^* + \boldsymbol{\epsilon}$.

- This is what we did when looking at the epidemic models at their onset

Local dynamics and stability

- We do a Taylor expansion of the dynamics around $\boldsymbol{x}^*$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon_i}{\mathrm{d}t} = f\left(x_i^* + \epsilon_i\right) + \sum_j A_{ij} g\left(x_i^* + \epsilon_i, x_j^* + \epsilon_j\right)$$

$$= f\left(x_i^*\right) + \epsilon_i \frac{\mathrm{d}f}{\mathrm{d}x}\bigg|_{x=x_i^*} + \sum_j A_{ij} g\left(x_i^*, x_j^*\right)$$

$$+ \epsilon_i \sum_j A_{ij} \frac{\partial g(u,v)}{\partial u}\bigg|_{u=x_i^*, v=x_j^*} + \sum_j A_{ij} \epsilon_j \frac{\partial g(u,v)}{\partial v}\bigg|_{u=x_i^*, v=x_j^*}$$

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Local dynamics and stability

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- Defining:

$$\alpha_i = \left.\frac{\partial f}{\partial x}\right|_{x=x_i^*}, \ \beta_{ij} = \left.\frac{\partial g(u,v)}{\partial u}\right|_{u=x_i^*, v=x_j^*}, \ \gamma_{ij} = \left.\frac{\partial g(u,v)}{\partial v}\right|_{u=x_i^*, v=x_j^*}$$

the equation becomes

$$\frac{d\epsilon_i}{dt} = \left[\alpha_i + \sum_j \beta_{ij} A_{ij}\right]\epsilon_i + \sum_j \gamma_{ij} A_{ij} \epsilon_j, \quad \Rightarrow \quad \dot{\boldsymbol{\epsilon}} \equiv \boldsymbol{M}\boldsymbol{\epsilon}$$

the matrix describing the local dynamics of the perturbation is:

$$[\boldsymbol{M}]_{ij} = \begin{cases} \alpha_i + \sum_k \beta_{ik} A_{ik} & \text{for } i = j \\ \gamma_{ij} A_{ij} \end{cases}$$

Analysis of local linear dynamics

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- The eigenvalues $\boldsymbol{M}$ determine if $\boldsymbol{\epsilon}(t)$ moves away from the fixed point, or if it dies out, and therefore the system goes back to $\boldsymbol{x}^*$
- The solution depends on the EVD of $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$. Calling $\varepsilon(0) = \boldsymbol{U}^{-1}\boldsymbol{\epsilon}(0)$, like in the previous cases:

$$\boldsymbol{x}(t) \approx \boldsymbol{x}^* + \sum_{k=1}^{N} \varepsilon_k(0) e^{\lambda_k t} \boldsymbol{u}_k$$

- Unless all $\lambda_k > 1$ the fixed point is unstable

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Two interesting cases

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- One case where the dependency is clearer is when $\boldsymbol{x}^* = x^*\mathbf{1}$, i.e. the fixed point has all identical coordinates because all the derivatives are the same (like for SI and SIR models at the onset, where $\boldsymbol{x}^* = \mathbf{0}$, which is (an unstable) fixed point.
- In this case:

$$\alpha_i = \alpha = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}, \beta_{ij} = \beta = \left.\frac{\partial g\left(u,v\right)}{\partial u}\right|_{u=x^*,v=x^*}, \gamma_{ij} = \gamma = \left.\frac{\partial g\left(u,v\right)}{\partial v}\right|_{u=x^*,v=x^*}$$

$$\Rightarrow \quad \boldsymbol{M} = \alpha\boldsymbol{I} + \beta\text{diag}(\boldsymbol{A}\mathbf{1}) + \gamma\boldsymbol{A}$$

Two interesting cases

- Because $\boldsymbol{M} = \alpha\boldsymbol{I} + \beta\boldsymbol{L}$ has eigenvalues $\alpha + \beta\lambda_k(\boldsymbol{L})$, stability of the fixed point is directly tied to the spectrum of $\boldsymbol{L}$

$$\alpha + \beta\lambda_k(\boldsymbol{L}) < 1$$

- Since $\lambda_N = 0$, it follows that for stability $\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \alpha < 1$
- For stability, if $\left.\frac{\partial g(u,v)}{\partial u}\right|_{u=x^*,v=x^*} = \beta$

$$\frac{1}{\lambda_1(\boldsymbol{L})} > -\frac{\alpha}{\beta}$$

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Affine linear dynamics

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- So far we have considered *closed network dynamics*, where nodal state variables (i.e. the graph signal) $\boldsymbol{x}(t)$ vary due to interactions with the neighboring state variables

- In reality, networks have external inputs and outputs that are exogenous

- Linear affine equations are in the discrete and ODE form respectively

$$\boldsymbol{x}(t+1) = \boldsymbol{W}\boldsymbol{x}(t) + \boldsymbol{b} \quad \text{discrete linear dynamics}$$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{M}\boldsymbol{x}(t) + \boldsymbol{b} \quad \text{continuous linear dynamics}$$

- Do they converge, and if so, where?

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Discrete Affine Linear dynamics

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- Note that if $(\boldsymbol{I}-\boldsymbol{W})$ is invertible, then the discrete dynamics $\boldsymbol{x}(t+1)=\boldsymbol{W}\boldsymbol{x}(t)+\boldsymbol{b}$ have the following fixed point:

$$\boldsymbol{x}^*=(\boldsymbol{I}-\boldsymbol{W})^{-1}\boldsymbol{b}$$

- In order for $(\boldsymbol{I}-\boldsymbol{W})$ to be invertible all the eigenvalues have to be non zero.
- Since the eigenvalues of $(\boldsymbol{I}-\boldsymbol{W})$ are $1-\lambda_k(\boldsymbol{W})$, this requires that no $\lambda_k(\boldsymbol{W})=1$
- Note that this is clearly not true for the average consensus matrix $\boldsymbol{W}=\boldsymbol{I}-\alpha\boldsymbol{L}$ and for any stochastic matrix in general!

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- Note that this is clearly not true for the average consensus matrix $\boldsymbol{W}=\boldsymbol{I}-\alpha\boldsymbol{L}$ and for any stochastic matrix in general!

Convergenge of Discrete Affine Linear dynamics

- Even if the matrix $(\boldsymbol{I} - \boldsymbol{W})$ is full rank, having a fixed point does not mean that the system will converge to that (as it could be unstable)

- It can be shown that the state at time t is:

$$\boldsymbol{x}(t) = \boldsymbol{W}^t \boldsymbol{x}(0) + \left(\sum_{k=0}^{t} \boldsymbol{W}^{t-k}\right) \boldsymbol{b}$$

- Note that this equation can be generalized to the case where the additive component is changing $\boldsymbol{b}(t)$

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Discrete Affine Linear dynamics

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- Note with a change of summation index $k' = t - k$ the matrix series can be written as $\sum_{k=0}^{t} \boldsymbol{W}^{t-k} \equiv \sum_{k'=0}^{t} \boldsymbol{W}^{k'}$
- The series converges as $t \to +\infty$ if and only if all the eigenvalues of $\boldsymbol{W}$ are strictly less then 1.
- In fact, let the EVD of $\boldsymbol{W} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$. Then

$$\sum_{k=0}^{t} \boldsymbol{W}^{k} = \boldsymbol{U}\left(\sum_{k=0}^{t} \boldsymbol{\Lambda}^{k}\right)\boldsymbol{U}^{-1}.$$

- Recall the geometric series is defined as $\sum_{k=0}^{+\infty} \rho^{k}$ and the series converges to $(1-\rho)^{-1}$ if and only if $|\rho| < 1$
- If all $|\lambda_i| < 1$ then series of each diagonal element of $\sum_{k=0}^{t} \boldsymbol{\Lambda}^{k}$ converges to $\sum_{k=0}^{+\infty} \lambda_i^{k} = (1-\lambda_i)^{-1}$.

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Discrete Affine Linear dynamics

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- This implies that if all $|\lambda_i| < 1$

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Discrete Affine Linear dynamics

Cornell University

Affine discrete dynamics

The affine discrete dynamics $\boldsymbol{x}$ converge to

$$\boldsymbol{x}^* = (\boldsymbol{I} - \boldsymbol{W})^{-1}\boldsymbol{b}$$

if and only if $\boldsymbol{W}$ has all eigenvalues <1. For a matrix with non-negative weights the matrix has to be **sub-stochastic**, i.e. $\mathbf{1}^\top \boldsymbol{W} < 1$.

- Note that the final state of the system does not depend at all on the initial state $\boldsymbol{x}(0)$
- The input $\boldsymbol{b}$, chosen appropriately knowing what $(\boldsymbol{I} - \boldsymbol{W})^{-1}$ is, can lead the nodes to any arbitrary state.
- We will use this fact in modeling opinion dynamics under the influence of external pressure

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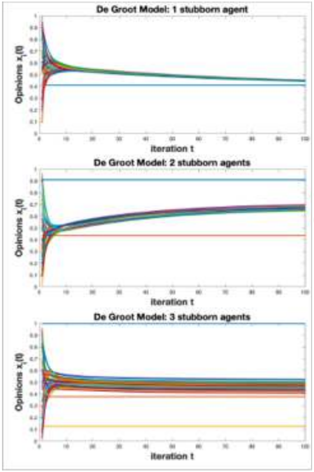
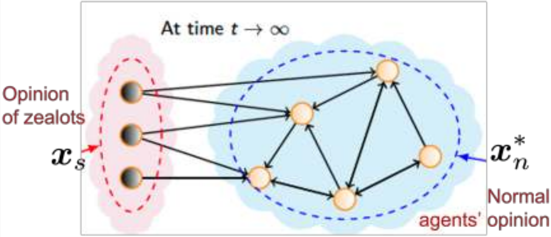
Example: Opinion dynamics with zealots





$$\boldsymbol{b} = \boldsymbol{A}_{sn}\boldsymbol{x}_s \qquad \boldsymbol{x}_n^* = (I - \boldsymbol{W})^{-1}\boldsymbol{A}_{ns}\boldsymbol{x}_s$$

More details will come when modeling social networks...

Continuous Affine Linear dynamics

- They are described by affine ODEs in the form:

$$\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{b}$$

- These ODEs convergence depends on the properties of $\boldsymbol{M}$. Metzler matrices characterize these systems.

Positive systems

Cornell University

Definition of Metzler matrices

An $N \times N$ square matrix $\boldsymbol{M}$ is a Metzler matrix if:

1. The off-diagonal elements of $\boldsymbol{M}$ are non-negative, i.e. $[\boldsymbol{M}]_{ij} \geq 0$ for $i \neq j$, the diagonal elements can be negative
2. $\boldsymbol{M}$ has an associated directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with edge weights $[\boldsymbol{M}]_{ij} > 0$ and with no self-loops
3. For any Metzler matrix, there exist a constant $c > 0$ such that $\boldsymbol{M} + c\boldsymbol{I}$ has all non negative eigenvalues

- An example of a Metzler matrix is the complement of the Laplacian $-\boldsymbol{L}$, whose dominant eigevalue is 0

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Positive systems: Metzler matrices

Cornell University

Properties of irreducible Metzler matrices

From Perron-Frobenious theorem:

1. There exist an eigenvalue $\lambda^* > \mathbb{R}\{\lambda_k(\boldsymbol{M})\}$ for all other k.
2. If the graph associated to $\boldsymbol{M}$ is strongly connected λ^* is unique and the right and left eigenvectors entries associated to λ^* are positive (up to rescaling).

- We are interested in this kind of matrices, because in flow networks we have affine linear ODE equations.

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Positive continuous affine linear dynamics

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- In particular, these systems models have $\boldsymbol{x}(t) \geq \mathbf{0}, \quad \forall t \geq 0$.

Positive affine systems

An affine system $\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{b}$ is positive if and only if $\boldsymbol{M}$ is Metzler and $\boldsymbol{b} \geq \mathbf{0}$, with initial condition $\boldsymbol{x}(0) \geq 0$, that is $\boldsymbol{x}(t) \geq 0$ for all $t > 0$.

- Also in this case we can see that there is a fixed point

$$\boldsymbol{x}^* = -\boldsymbol{M}^{-1}\boldsymbol{b}, \quad \dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x}^* + \boldsymbol{b} = -\boldsymbol{M}\boldsymbol{M}^{-1}\boldsymbol{b} + \boldsymbol{b} = -\boldsymbol{b} + \boldsymbol{b} = \mathbf{0}$$

  and because at $\boldsymbol{x}^* = M^{-1}\boldsymbol{b}$ the derivative does not change, that is a fixed point.

- Like before, we need to understand when the system convergences to it

Positive systems: Metzler matrices

Cornell University

Hurwitz matrix

$\boldsymbol{M}$ is Hurwitz if all its eigenvalues have negative real part, i.e.

$$\mathbb{R}\{\lambda_k\} < 0 \quad \forall k = 1, N$$

- For such matrices, the homogeneous linear dynamics (i.e. $\boldsymbol{b} = \boldsymbol{0}$) converge to zero from any initial condition
- In fact, let $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$, and $\boldsymbol{\xi}(t) = \boldsymbol{U}^{-1}\boldsymbol{x}(t)$

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Linear dynamics

Cornell University

- The ODEs $\dot{\boldsymbol{\xi}} = \Lambda\boldsymbol{\xi}$ on the entries of $\boldsymbol{\xi}$ are decoupled, since $\boldsymbol{\Lambda}$ is diagonal, and for each entry the solution is:

$$\dot{\xi}_k = \lambda_k \xi_k \quad \Rightarrow \quad \xi_k(t) = \xi_k(0)e^{\lambda_k t}$$

$$\Rightarrow \quad \boldsymbol{x}(t) = \sum_{k=1}^{N} \xi_k(0)e^{\lambda_k t}\boldsymbol{u}_k \; \Rightarrow \; \lim_{t\to+\infty} \boldsymbol{x}(t) = \mathbf{0} \;\text{ if }\; \mathbb{R}\{\lambda_k\} < 0$$

- Note that a more compact way of writing this equation is to leverage a matrix exponential

$$e^{\boldsymbol{M}t} = \boldsymbol{U}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & \dots & \dots & e^{\lambda_N t}\boldsymbol{U}^{-1} \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}(t) = e^{\boldsymbol{M}t}\boldsymbol{x}(0)$$

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Solution

- Let $\boldsymbol{\beta} = \boldsymbol{U}^{-1}\boldsymbol{b}$. The solution of the affine dynamics can be obtained similarly, by solving the individual scalar equations: $\dot{\xi}_k = \lambda_k \xi_k + \beta_k$.
- The solution is obtained by using the Laplace transform, and is, for a time varying $\beta_k(t)$ as follows

$$\xi_k(t) = \xi_k(0)e^{\lambda_k t} + \int_0^t e^{\lambda_k(t-\tau)}\beta_k(\tau)d\tau = \xi_k(0)e^{\lambda_k t} + \overbrace{e^{\lambda_k t} \star \beta_k(t)}^{\text{convolution}}$$

- When $\beta_k(t) = \beta_k$, then:

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Convergence with Hurwitz matrix

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Positive systems: Hurwitz-Metzler matrices

Inverse positivity: Hurwitz -Metzler matrices

A Metzler matrix is Hurwitz-Metzler if $\forall \boldsymbol{b} \geq \mathbf{0}$, there exist a unique positive solution $\boldsymbol{x}^*$ to $\boldsymbol{M}\boldsymbol{x}^* + \boldsymbol{b} = \mathbf{0}$. A Hurwitz Metzler matrix associated to a strongly connected graph also has $-\boldsymbol{M}^{-1}$ which is positive definite.

- In this case, as long as $\boldsymbol{b} \geq \mathbf{0}$ the fixed point and any intermediate value is non-negative:

$$t \gg 1 \quad \boldsymbol{x}(t) = e^{\boldsymbol{M}t}\boldsymbol{x}(0) + \left(\int_0^t e^{\boldsymbol{M}(t-\tau)} d\tau\right)\boldsymbol{b} \;\rightarrow\; -\boldsymbol{M}^{-1}\boldsymbol{b} \geq \mathbf{0}$$

Outline

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Dynamical Flow Systems

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- Flow Systems are dynamical systems with state variables that are non-negative, and characterized by conservation laws (mass, energy)
- **Examples**: transportation networks, queueing networks, communication networks, some ecological and biological networks.

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Dynamical flow systems

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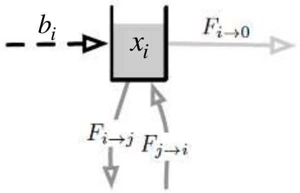
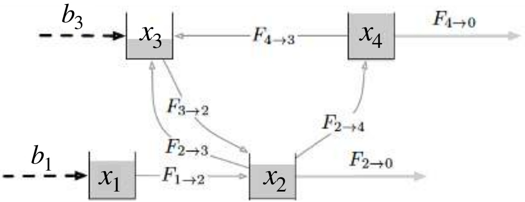
- Dynamical flow systems modeling requires understanding the dynamics of these affine systems
- They are systems in which *mass* is stored at nodes and can be transferred over a directed graph from one buffer to the other
- Let us denote the flow directed from node i to node j in the network as $F_{i\to j}$, and denote by $F_{i\to 0}$ the *outflows* that exit the network and b_i the input to each node



(a) A compartment with inflow b_i, outflow $F_{i\to 0}$, and inter-compartmental flows $F_{i\to j}$



(b) A compartmental system with two inflows and two outflows

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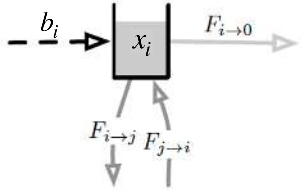

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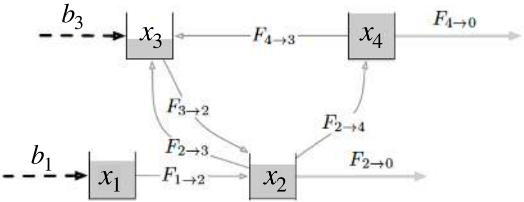
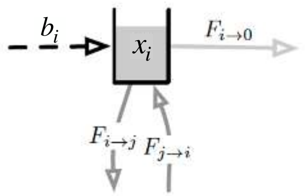


(b) A compartmental system with two inflows and two outflows

Dynamical flow systems

Cornell University

- Dynamical flow systems modeling requires understanding the dynamics of these affine systems
- They are systems in which *mass* is stored at nodes and can be transferred over a directed graph from one buffer to the other
- Let us denote the flow directed from node i to node j in the network as $F_{i \to j}$, and denote by $F_{i \to 0}$ the *outflows* that exit the network and b_i the input to each node



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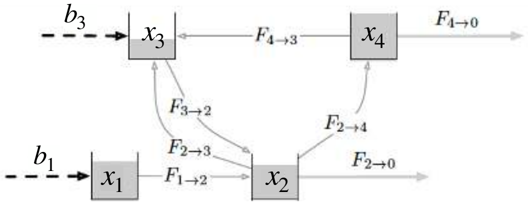

(b) A compartmental system with two inflows and two outflows

Dynamical flow systems equations

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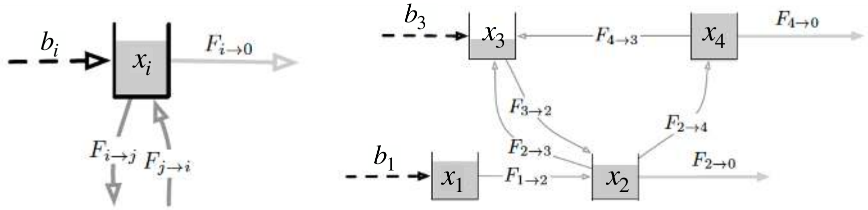

(a) A compartment with inflow b_i, outflow $F_{i\to 0}$, and inter-compartmental flows $F_{i\to j}$

(b) A compartmental system with two inflows and two outflows

- The state of each node is $x_i(t)$; it is replenished by the input $b_i \geq 0$ and depleted by the flow $F_{i\to 0}(x_i)$
- In general, the flow depends on the state of the buffer, i.e. $F_{i\to j} = F_{i\to j}(x_i)$ and the input b_i could also depend on it, but usually it is the outflow. Specifically, flow functions must satisfy:

$$F_{i\to j}(x) \geq 0, \quad F_{i\to j}(0) = 0$$

Dynamical flow systems equations

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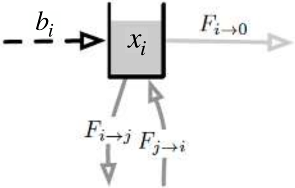

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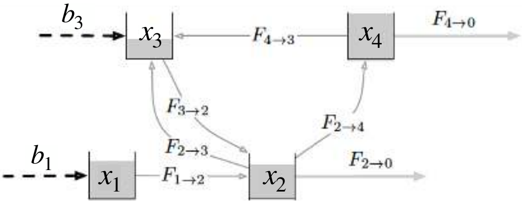

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Dynamical flow systems equations

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- The dynamics of the buffer state are:

$$\dot{x}_i = \overbrace{b_i - F_{i\to 0}(x_i)}^{f(x_i)} + \overbrace{\sum_{j\in\mathcal{N}_i^{\text{in}}} F_{j\to i}(x_j) - \sum_{j\in\mathcal{N}_i^{\text{out}}} F_{i\to j}(x_i)}^{\sum_{j\in\mathcal{N}_i} g(x_i,x_j)}$$

- Let B be the oriented incidence matrix. With $[\boldsymbol{f}(\boldsymbol{x})]_{i\to j} = F_{i\to j}(x_i)$ for all $e = (i \to j) \in \mathcal{E}$ for $j \neq 0$, $[\mathbf{f}_{\text{out}}]_i = F_{i\to 0}$ and $[\boldsymbol{b}]_i = b_i$

$$\Rightarrow \quad \dot{\boldsymbol{x}} = \boldsymbol{b} - \mathbf{f}_{\text{out}}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{x})$$

- The vector $\boldsymbol{b}$ is called the **in-flow** and $\mathbf{f}_{\text{out}}$ the **out-flow**
- The flows $\boldsymbol{f}(\boldsymbol{x})$ are non negative and are the outflows of each node to each of its out-neighbors

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Conservation of mass

- Note that the solution of the equations is consistent with the **conservation of mass**:

$$\mathbf{1}^\top \dot{\boldsymbol{x}} = \mathbf{1}^\top \boldsymbol{b} - \mathbf{1}^\top \mathbf{f}_{\text{out}}(\boldsymbol{x}) + \overbrace{\mathbf{1}^\top \boldsymbol{B} \boldsymbol{f}(\boldsymbol{x})}^{=0}$$

  recall $\mathbf{1}^\top \boldsymbol{B} = \mathbf{0}$ always for the oriented incidence matrix.

- Note that $\mathbf{1}^\top \dot{\boldsymbol{x}}$ is the cumulative change in mass in the network
- The equation says it is equal to $(\mathbf{1}^\top \boldsymbol{b} - \mathbf{1}^\top \mathbf{f}_{\text{out}}(\boldsymbol{x}))$, which is the difference of the cumulative in and out flows
- If $(\mathbf{1}^\top \boldsymbol{b} - \mathbf{1}^\top \mathbf{f}_{\text{out}}(\boldsymbol{x})) = 0$ the total mass $\mathbf{1}^\top \boldsymbol{x}$ is conserved, otherwise it increases or decreases.

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Special case: Linear flow functions

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- We note that these are non linear dynamics like the ones we defined in general:
  1. The node state x_i is the graph node signal x_i
  2. The local function $f_i(x)$ is $(b_i - F_{i\to 0}(x))$
  3. The edge function $g_{ij}(x_i, x_j)$ is $(F_{j\to i}(x_j) - F_{i\to j}(x_i))$ is
- A special case is when the functions are such that:

$$F_{i\to j}(x_i) = w_{ji}x_i \propto x_i \quad \text{outflow from } i \text{ to } j \text{ proportional to } x_i$$

$$F_{i\to 0}(x_i) = w_{0i}x_i \propto x_i \quad \text{external outflow proportional to } x_i$$

$$b_i \quad \text{is not a function of the queue state}$$

Linear flow functions dynamics

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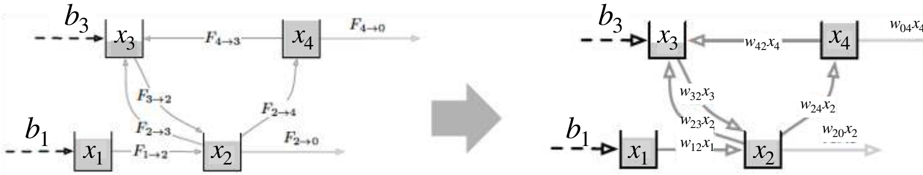

- In this special case:

$$\dot{x}_i = -\left(w_{0i} + \sum_{j \in \mathcal{N}_i^{\text{out}}} w_{ji}\right) x_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} w_{ij} x_j + b_i$$

Special case: Linear flow functions

- Introducing:
  1. the weighted adjacency matrix containing all flow rates $[\boldsymbol{A}]_{ij} = w_{ij}, i, j \in \mathcal{V}, \quad i \neq j$ and such that $[\boldsymbol{A}]_{ii} = 0$
  2. The out-degree weight vector $\boldsymbol{d}_{\text{out}} = \boldsymbol{A}\boldsymbol{1}$ (i.e. $[\boldsymbol{d}_{\text{out}}]_i = \sum_{j \in \mathcal{N}_i} w_{ij}$)
  3. The out-flow rate vector $[\boldsymbol{w}_o]_i = w_{i0}$
- The equation can be written as a function of the Laplacian matrix $\boldsymbol{L} = \text{diag}(\boldsymbol{d}_{\text{out}}) - \boldsymbol{A}$ as follows:

$$\dot{\boldsymbol{x}} = -(\text{diag}(\boldsymbol{w}_o) + \boldsymbol{L})\boldsymbol{x} + \boldsymbol{b}$$

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Special case: Linear flow functions

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- The matrix $\boldsymbol{M}$ where $\boldsymbol{M} = -(\text{diag}(\boldsymbol{w}_o) + \boldsymbol{L})$ is called the **compartmental matrix**
- In terms of the dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{b}$$

- Compartimental matrices are Metzler matrices:
  1. The off diagonal entries are non negative $M_{ij} > 0$
  2. The column sums are non positive $\mathbf{1}^\top \boldsymbol{M} =\leq 0$. In fact:

$$\begin{aligned}
\mathbf{1}^\top \boldsymbol{M} &= -\mathbf{1}^\top (\text{diag}(\boldsymbol{w}_o) + \boldsymbol{L}^\top) \\
&= -\mathbf{1}^\top \text{diag}(\boldsymbol{w}_o) - \mathbf{1}^\top \boldsymbol{L}^\top = -\boldsymbol{w}_o - \overbrace{\mathbf{1}^\top \boldsymbol{L}^\top}^{=\mathbf{0}} \\
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Outline

Concluding remarks

- In this lecture we discussed
  - General non-linear ODEs
  - Affine dynamics, continuous and discrete
  - Flow dynamics (I will expand briefly with some special cases)
- Next we will show the relationship between these abstract models in specific applications