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Lecture 8- Random Graph Models

Graph-Based Data Science For Networked Systems
ECE 5260 – ORIE 5735

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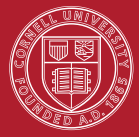
Cornell Tech

February 23, 2026

Content



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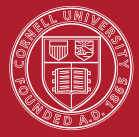
Outline



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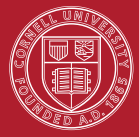
Introduction



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- We defined local (nodal) and group properties:
  1. Degree distribution
  2. Centrality metrics
  3. Similarity metrics and assortative networks
  4. Clustering coefficient
  5. Reciprocity
- These are helpful graph features to understand roles and special relationships in a network and predict what other neighbors can be to complete a network
- In this lecture we also want to introduce random graphs

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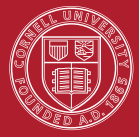

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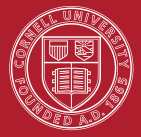

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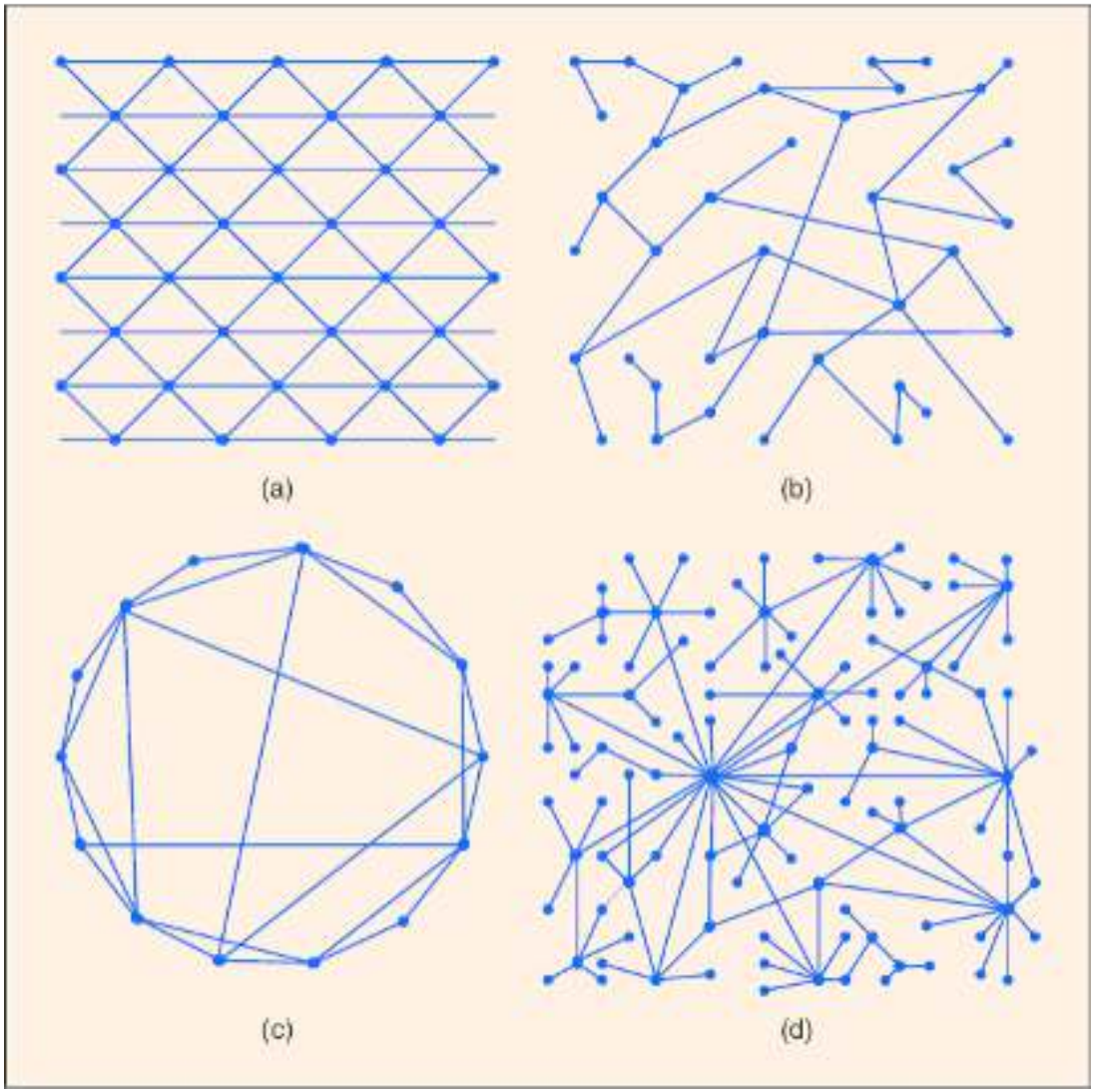
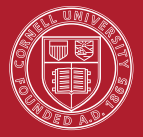
What is a random graph model?



- In any random experiments we need to define a sample space and a probability measure
- For random graphs this sample space is a set of graphs which makes this set discrete.

Why is that useful?



- Random graphs are abstractions of reality
- They show the idiosycracies of real graphs somewhat to the extreme
- They are a useful benchmarks for the properties of true networks
- A reasonably good fit helps making predictions.

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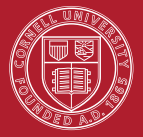

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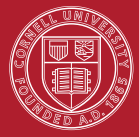

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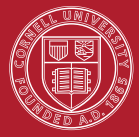

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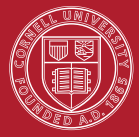

- We start considering *undirected graphs* and we will continue with them until otherwise stated.
- In our notation $P(A)$ will denote the **probability of the event** A, which includes a set of possible outcomes.
- $P(A|B)$ is the conditional probability of the event A given that the event B happened. Bayes law says:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

where $P(A \cap B)$ is the probability of the event that any outcome A and B have in common happened.

Random Graph probability

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- Typically, one either fixes the number of edges $M = |\mathcal{E}|$ or then number of nodes $N = |\mathcal{V}|$ or both.
- When both N and M are fixed then the graph density is fixed for all possible graph outcomes, and is:

$$\delta = \frac{|\mathcal{E}|}{\binom{N}{2}} = \frac{2M}{N(N-1)}$$

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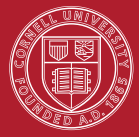

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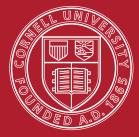
Random Graph: the uniform model

- The name Random Graph is often associated to a **specific model**: a graph obtained choosing uniformly at random the M pairs from $\mathcal{V}^2$ that form $\mathcal{E}$ such that $|\mathcal{E}| = M$
- The assumption specifies completely the random model, which we refer to using the notation $\mathcal{G}(N, M)$. In fact:
  1. The sample space contains all possible graphs
  2. The probability of each graph is uniform, so each of the graphs has probability equal to $1/|\mathcal{G}(N, M)|$

Random Graph probability



- There are $\binom{N}{2}$ possible edges.
- We need to chose M arbitrary combinations of them:

$$|\mathcal{G}(N,M)| = \#\text{ possible way of choosing } M \text{ elements our of } \binom{N}{2}$$

- This implies:

$$|\mathcal{G}(N,M)| = \binom{\binom{N}{2}}{M} \quad \Rightarrow \quad P(\mathcal{G}) = \frac{1}{\binom{\binom{N}{2}}{M}}$$

- You could interpret this as generating the $N \times M$ incidence matrix B unifromly at random

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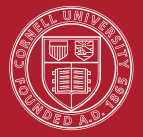

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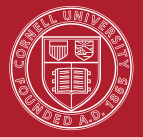

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Random Graph: averages

- If we consider a graph metric $\phi(\mathcal{G})$, an interesting analysis is what is the average across all these graph samples using the graph distribution $P(\mathcal{G})$

$$\mathbb{E}[\phi(\mathcal{G})] = \sum_{\mathcal{G} \in \mathcal{G}(N,M)} \ell(\mathcal{G}) \qquad P(\mathcal{G}) = \frac{1}{\binom{\binom{N}{2}}{M}} \sum_{\mathcal{G} \in \mathcal{G}(N,M)} \phi(\mathcal{G})$$

- When $N \gg 1$ and $M \gg 1$ these quantities tend to **harden**, i.e. go to a limit that depend on graph density because of the law of large numbers

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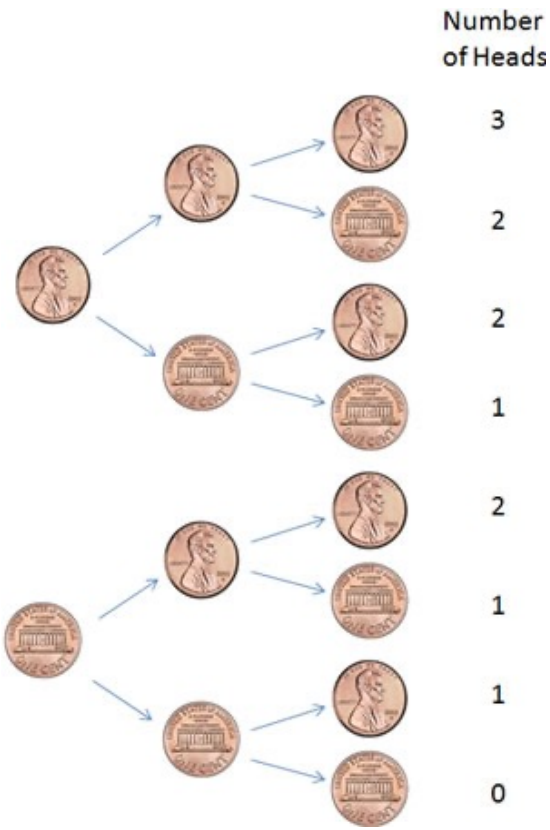
Poisson/Erdös and Rényi graphs

graph analysis

- Fixing the number of edges makes computing expectations and limits of these random graphs hard
- A more tractable model is one where the probability p that two nodes are connected is fixed, but not how many edges are in $\mathcal{E}$.
- This model is referred to as the **Poisson Random Graphs** or **Erdös and Rényi graphs**, the notation we use is $\mathcal{G}(N, p)$
- One can interpret each graph formation as a Binomial experiment that picks a pair among then $\binom{N}{2}$ possible with probabilty p.



If the coin is "Head" there is an edge between $(u, v) \in \mathcal{V} \times \mathcal{V}$ with $u \neq v$, if it is "Tail" then u and v are not neighbors.

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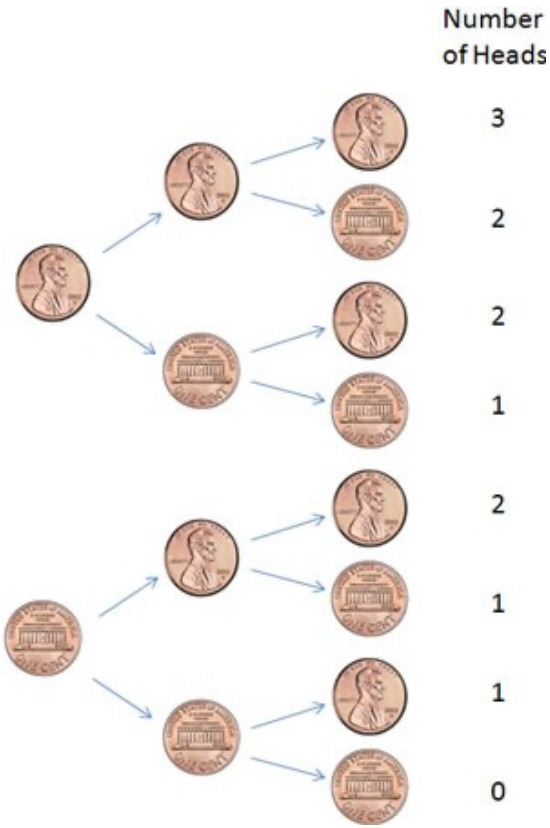

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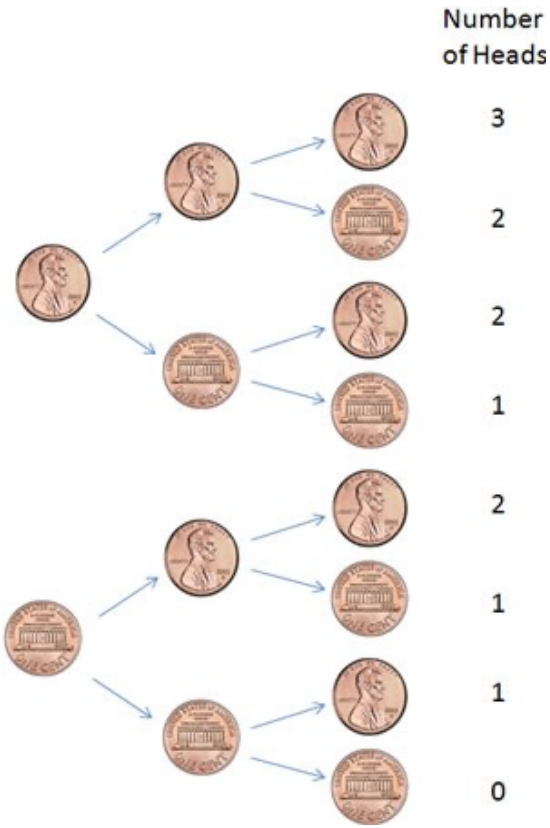

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- Let us indicate by $X \sim \mathcal{B}(K, p)$ a random variable whose distribution is Binomial with parameter p (probability of success) and K (number of trials).

$$p_k \triangleq P(X = k) = \binom{K}{k} p^k (1-p)^{K-k}$$

- This is the probability of having any of the sequences with k successes. But the probability of one specific sequence is $p^k(1-p)^{K-k}$.
- The probability of a specific graphs with M edges is the same and is the probability of M successes (links) over $\binom{N}{2}$ possibilities:

$$P(\mathcal{G}||\mathcal{E}| = M) = p^M (1-p)^{\binom{N}{2}-M}$$

in other words, $|\mathcal{E}| \sim \mathcal{B}(\binom{N}{2}, p)$ and all graphs with the same number of edges have equal probabilty $p^M(1-p)^{\binom{N}{2}-M}$.

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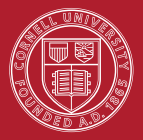

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Erdös and Rényi graphs averages

- This model greatly simplifies several calculation of average graph features
- Average number of edges: It is the mean of the Binomial random variable that defines the graph probability
- The expected value for the Binomial distribution $\mathcal{B}(K, p)$ is $\mathbb{E}[X] = Kp$.
- Therefore, given that there are $\binom{N}{2}$ possible edges, the expected number of edges of an Erdös and Rényi graph $\mathcal{G}(N, p)$ is:

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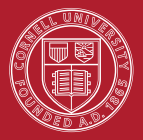

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- From the average number of edges we can easily obtain the average degree
- From the Handshaking theorem, the average degree is

$$\frac{1}{N}\sum_{v\in\mathcal{V}} d_v = 2\frac{|\mathcal{E}|}{N}$$

Hence, recalling that $\binom{N}{2} = \frac{1}{2}N(N-1)$, averaging over the distribution of $\mathcal{G}(N,p)$ the average degree gives:

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Degree distribution for $\mathcal{G}(N,p)$

graph analysis

- The observation is interesting because it is clear that in order to have the expected degree in the limit as $N \to +\infty$ coverge to a finite value > 0, p cannot vanish faster than $O(1/N)$
- For $\mathcal{G}(N,p)$ we can actually specify the entire degree distribution, which is:

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

- This is again a Binomial distribution, but in this case is $\mathcal{B}(N-1,p)$. So $\mathbb{E}[d_v] = (N-1)p$.
- The reason is that the edges connecting each node to the remaining $N-1$ is like the Binomial experiment of $N-1$ trials with probabilty of success (i.e. of an edge) equal to p

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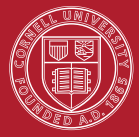

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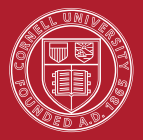
Asymptotic analysis



- The Binomial distribution has two possible limits, for $N \to +\infty$:
  1. the Gaussian distribution if p is constant, i.e. $d_v \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = pN$ and $\sigma^2 = p(1-p)N$
  2. the Poisson distribution when $p = \frac{c}{N-1} \to 0$
- The trend for large $\mathcal{G}(N,p)$ networks has been widely studied, considering the "Poisson case" in which the limit of the degree is a constant, which means

$$p = \frac{c}{N-1}, \quad 0 < c < +\infty \quad \text{a constant}$$

- Note that $\to \mathbb{E}[d_v] = c$

Asymptotic analysis

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- The trend for large $\mathcal{G}(N,p)$ networks has been widely studied, considering the "Poisson case" in which the limit of the degree is a constant, which means

$$p = \frac{c}{N-1}, \quad 0 < c < +\infty \quad \text{a constant}$$

- Note that $\to \mathbb{E}[d_v] = c$

Asymptotic analysis

- The Binomial distribution has two possible limits, for $N \to +\infty$:
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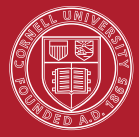

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- Specifically, the Poisson distribution is:

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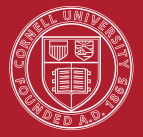

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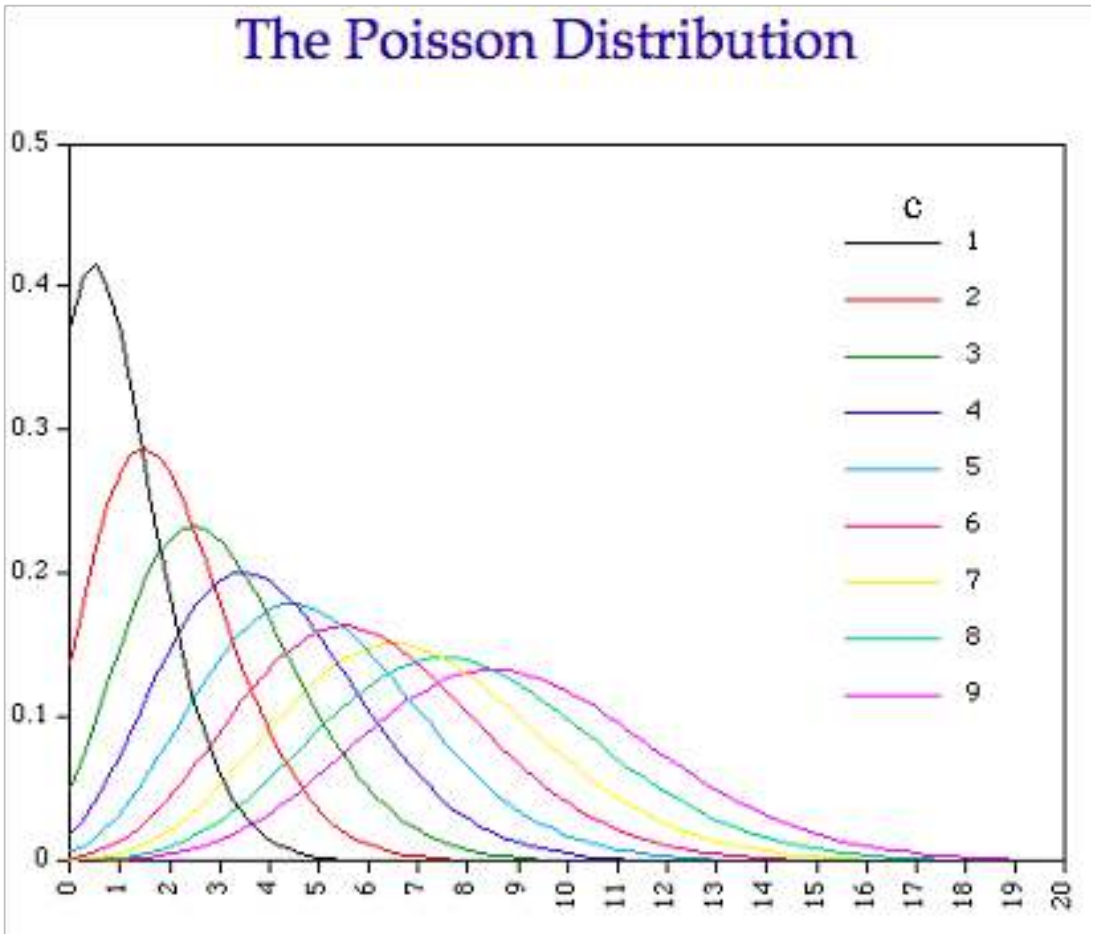
Degree distribution shape

The Poisson distribution has exponentially decaying tails. It is unimodal with mode c



Clustering coefficient of $\mathcal{G}(N,p)$

graph analysis

- Recall that the clustering coefficient measures the transitivity of nodes with respect of being connected
- The probability of closing three nodes that are connected in a cycle is the same as the probability of adding a third edge, which is p. Then

$$\mathbb{E}[C] = p$$

- In the asymptotic regime, this means that $C = \frac{c}{N-1} \to 0$
- This is one of the most striking differences of $\mathcal{G}(N,p)$ with respect to, for instance, social networks, where friends of friends tend to be friends.

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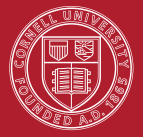

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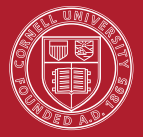

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Connectivity and Giant Component

- A giant component is a graph component that includes most of the nodes N. Let us call the set of nodes in the giant component $\mathcal{V}^g \subseteq \mathcal{V}$
- Is there a value of p that leads with high probability to the emergence of this giant components in all possible outcomes of $\mathcal{G}(N, p)$?
- Yes. This value can be calculated if we study the expected fraction of nodes γ that are in the giant componet, i.e.

$$\gamma := \frac{\mathbb{E}[|\mathcal{V}^g|]}{N} \equiv P(\text{a randomly chosen node } v \in \mathcal{V}^g)$$

and so $1 - \gamma = P(\text{a randomly chosen node } u \notin \mathcal{V}^g)$

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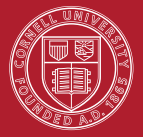

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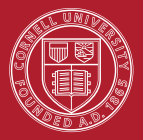

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Probability v not in the giant component

- The probability that a node v is not in $\mathcal{V}^g$ is equal to the probability of the event that v has no edge connecting it to a nodes in $\mathcal{V}^g$
- The probability that v has no edge connecting it to a nodes in $\mathcal{V}^g$ can be decomposed in the occurrence of two events $\forall u \neq v$:
  1. **Event 1**: u is not connected to node v, so it does not count. This event has probablity $1-p$
  2. **Event 2**: $u \notin \mathcal{V}^g$ and it is connected to node v, and $v \notin \mathcal{V}^g$. This event has probability $(1-\gamma)p$
- The events are mutually exclusive, so we can add their respective probabilities to obtain:

$$P(u \neq v \text{ Event 1 happens or Event 2 happens}) = 1 - p + (1-\gamma)p$$

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- From this we can derive the following equation:

$$
\begin{aligned}
1-\gamma &= P(\text{a randomly chosen node } v \notin \mathcal{V}^g) \\
&= P(\forall u \neq v \text{ Event 1 happens or Event 2 happens}) \\
&= \prod_{u \in \mathcal{V}, u \neq v} P(u \neq v \text{ Event 1 happens or Event 2 happens}) \\
&= (1-p+(1-\gamma)p)^{N-1} \\
&= (1+(1-\gamma-1)p)^{N-1} = (1-\gamma p)^{N-1}
\end{aligned}
$$

where the third equality is because the events for each node $u \neq v$ are independent and the next equation uses the expression we found in the previous slide, simplified in the last equation

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- With some further algebra, substituting $p = \frac{c}{N-1}$ we have:

$$N \gg 1 \quad \Rightarrow \quad 1 - \gamma \approx e^{-c\gamma} \quad \Rightarrow \quad \gamma \approx 1 - e^{-c\gamma}$$

- We cannot solve for γ explicitly, but it is relatively simple to get a numerical solution to find γ such that $\gamma \approx 1 - e^{-c\gamma}$
- Note that the solution is $\gamma = 0$ iff

$$1 = d(1 - e^{-c\gamma})/d\gamma|_{\gamma=0} = ce^{-c\gamma}|_{\gamma=0} = c$$

- This implies that there is a **phase transition** for $c \leq 1$ there is no giant component, while there is always a giant component, i.e. a component with $O(N)$ nodes for $c > 1$

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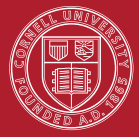

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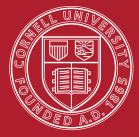

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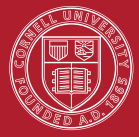

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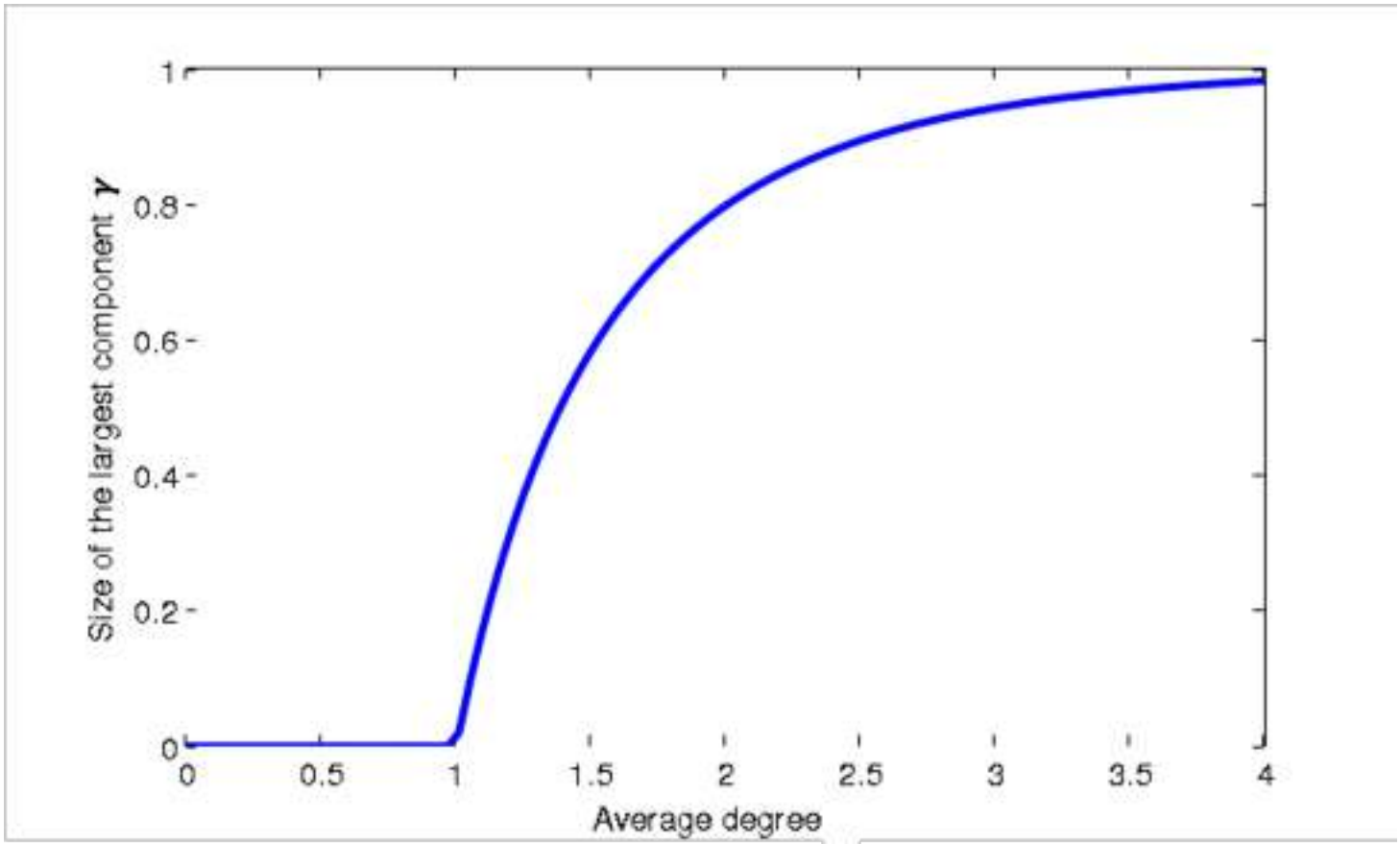
Phase transition

- The following figure illustrates the trend of the percentage size of the giant component γ versus the average degree c



Average path length

- A non-rigorous argument to compute the average path length works as follows:
  - Imagine we look at the formation of the network steps by step, starting from connecting one node at random, then a neighbor, then one this neighbor neighbors etc.
  - Recall $\mathbb{E}[d_v] = (N-1)p$ edges. For $p = \frac{c}{N-1}$ we are adding $\mathbb{E}[d_v] = c$ edges on average for each node.
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Average path length

- This trend must stop when $c^s = N$ because there are N nodes in total in the network!
- Hence, taking the natural logarithm at both sides of the equality $c^s = N$ we get:

$$s \approx \frac{\ln N}{\ln c}$$

- **Exact calculation**: Let dist_{uv} be the shortest path between nodes u and v. A more rigorous calculation for $N \gg 1$ (I will skip the details for brevity) yields the following result:

$$P(\text{dist}_{uv} > \ell) = \left(1 - \frac{c}{N}\right)^{c^{\ell-1}} \to e^{-\frac{c^\ell}{N}}$$

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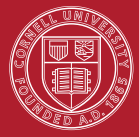

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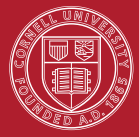

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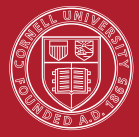

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How realistic is $\mathcal{G}(N, p)$?

- We have seen that $\mathcal{G}(N, p)$ allows for interesting and tractable analysis of the graph features
- One question is: "How realistic is $\mathcal{G}(N, p)$?"
- As a matter of fact, not that much, in most cases. Networks have more structure, even though it may be very different from one case to the other.
- This is readily apparent from comparing features such as the degree distribution
- The only feature whose trend is realistic is actually the average path length, which often tends to be in the order of $\ln N$

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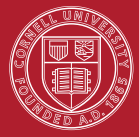

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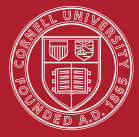

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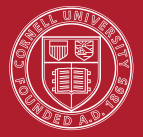

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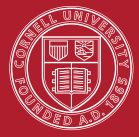
How realistic is $\mathcal{G}(N, p)$?



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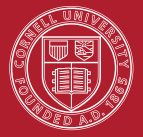

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- Empirical evidence shows that $\mathcal{G}(N,p)$ and some important networks of interest are very different
- The degree distribution for the Poisson model decays exponentially and drops very fast in log-log scale.
- **Scale free networks**: They are very frequent cases. "Scale free" networks instead have a degree distribution that is a *power law*, i.e. $p_k = \alpha k^{-p} + \beta$.
- In log-log scale it is a straight line. In fact, for a scale-free network we can write

$$y_k = \log p_k = -\log\alpha + p \overbrace{\log k}^{x_k \text{ in log-log scale}} + \log\beta$$

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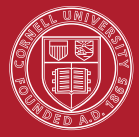

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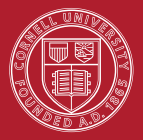

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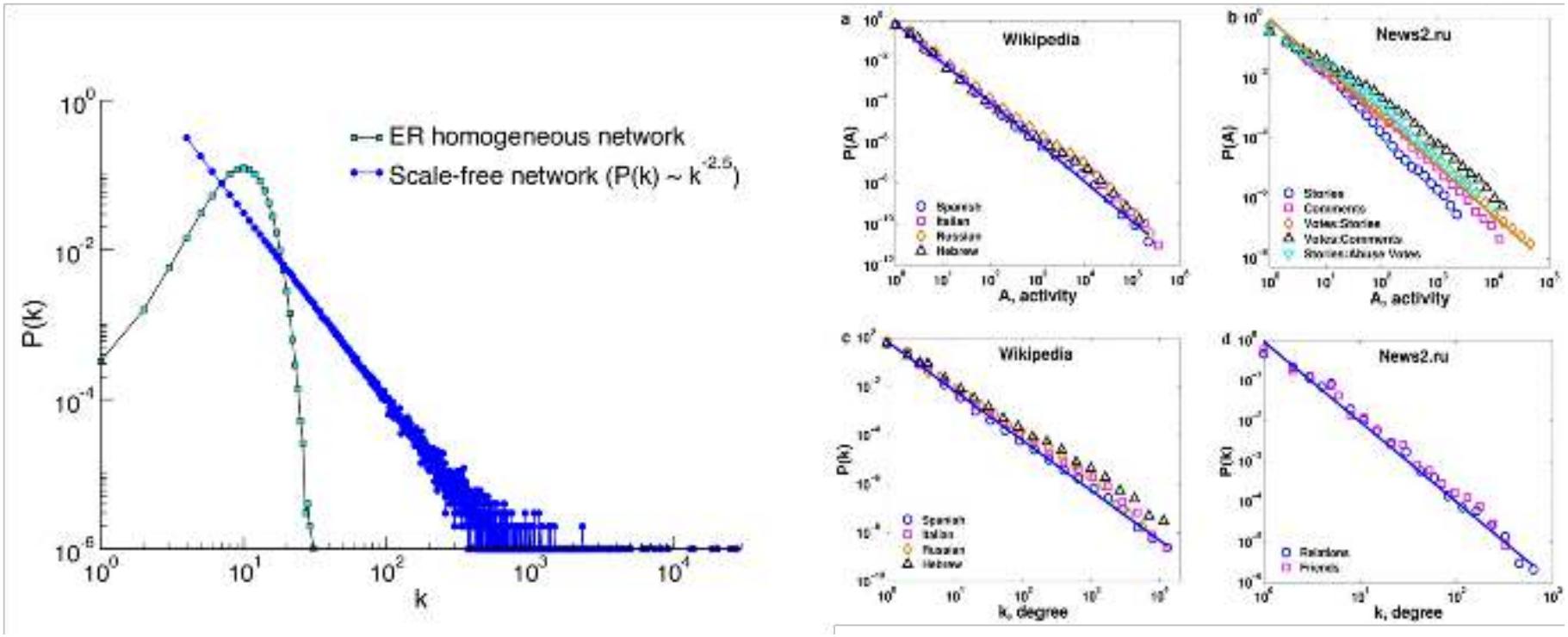
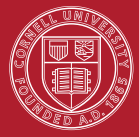
Comparison of $\mathcal{G}(N,p)$ features and real networks

- On the right we see a comparison of degree distributions of network and that of $\mathcal{G}(N,p)$ with a graph that has degree distribution that is a power law (i.e. it is "scale free")
- On the left we see the degree distribution in semi-logarithmic scale for Wikipedia in different countries



Outline



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The configuration model

graph analysis

- The basic idea is to fit an arbitrary degree sequence or degree distribution to the sample space of graphs
- There are two versions:
  1. In one the degree sequence is fixed d
  2. In the second one, the degree distribution p_k is given, and the degrees are drawn at random
- This model allows, for example, to match the power law trends we visualized in the previous slides

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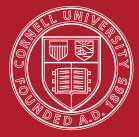

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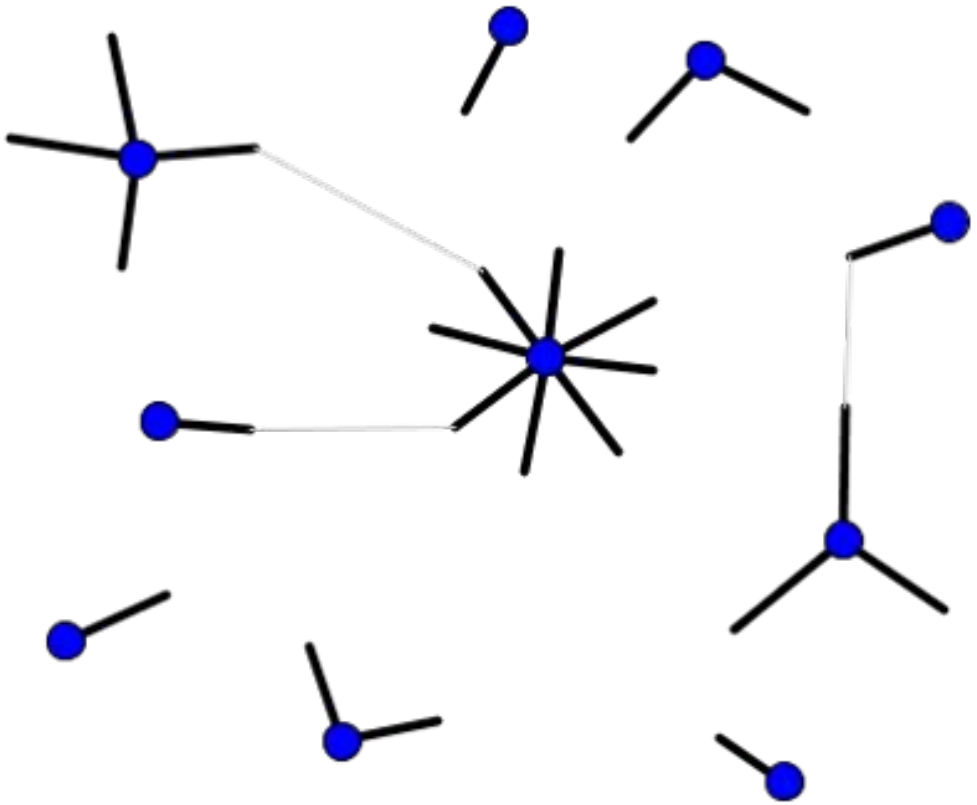
Configuration model: graph generation

graph analysis



- The N nodes are assigned the degrees, forming a series of stubs
- The second step is to match the stubs connecting each time one of the edges
- The matching stubs are chosen uniformly at random, in the set of those that have yet to be matched

Configuration model: graph generation

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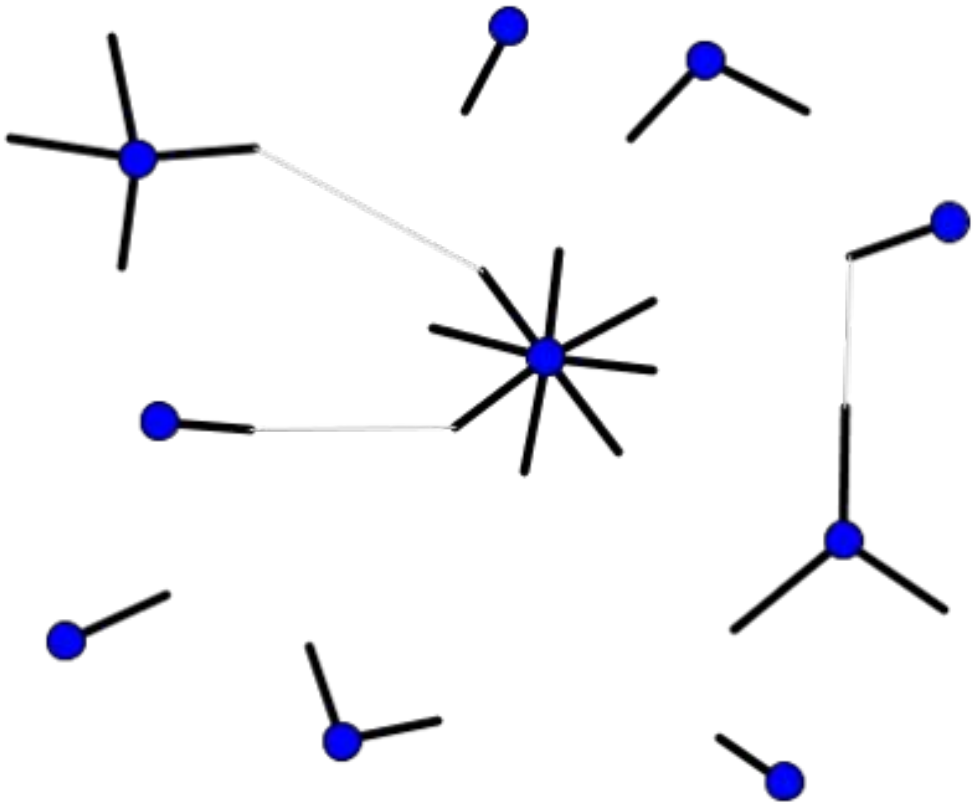

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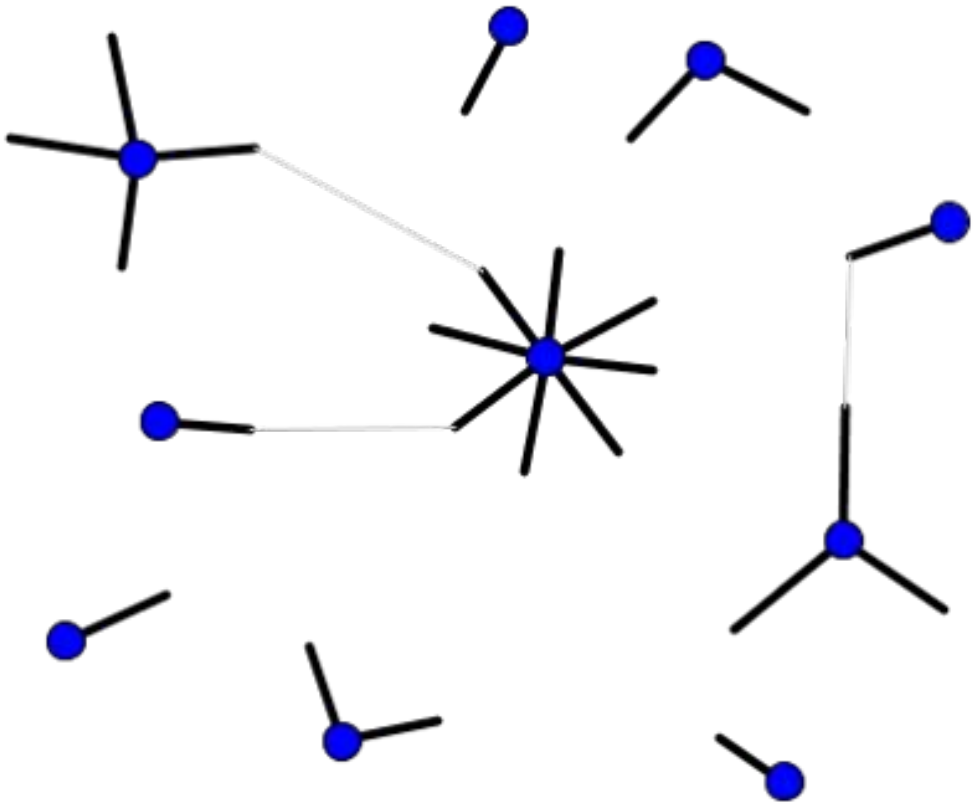

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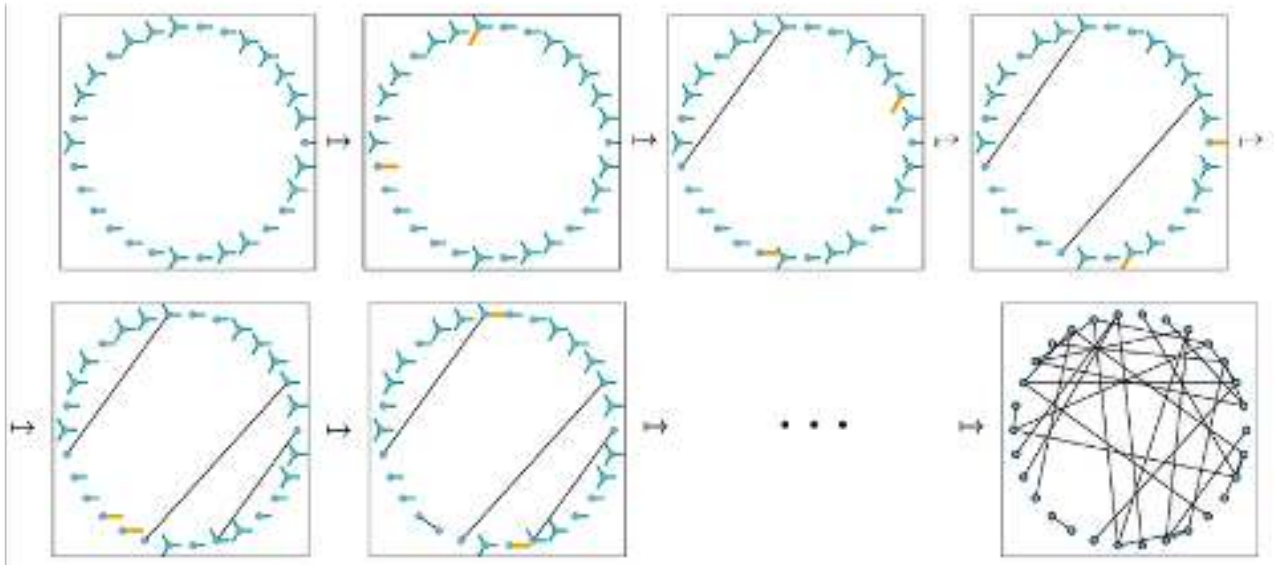
Forming the random graph



- This is an iterative process that can work **only** if the sum of the degree sequence is even, because if the handshaking theorem, i.e. $\boldsymbol{d}^\top \mathbf{1} = 2|\mathcal{E}|$
- If the degree sequence is drawn at random, we can throw away those that do not meet this criterion

Edge probability

- Recall $M = |\mathcal{E}|$, so there are $2M$ stubs total to match
- The probability of a match between node u stub with a specific edge node v stub chosen at random uniformly is $d_v/2M - 1$
- Adding the probability of the events that correspond to all possible d_u choices for the edge, the probablility of an edge between u and v is:

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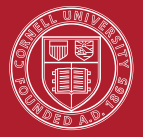
Multiedges and self-edges



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- Nothing prevents from choosing node v again to match another edge of node u and creating a multiedge
- Since one is already occupied, the probability of a second edge bewteen (u, v) is:

$$p_{uv}^{(2)} = \frac{d_u d_v}{2M-1} \times \frac{(d_u-1)(d_v-1)}{2M-1} = O(M^{-2})$$

  which is relatively small for large M relative to the first one which is $O(M^{-1})$
- It can also be shown that the probability of a self loop is:

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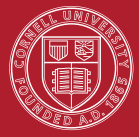

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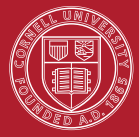
Friendship paradox



- One peculiar property of the configuration model is the so called *friendship paradox*: the degree of a node tends to be less than the average degree of its neighbors.
- Even though the configuration model does not hit necessarily the exact number, this phenomenon is observed in real networks

Network	n	Average degree	Average neighbor degree	$\frac{\langle k^2 \rangle}{\langle k \rangle}$
Biologists	1 520 252	15.5	68.4	130.2
Mathematicians	253 339	3.9	9.5	13.2
Internet	22 963	4.2	224.3	261.5

Friendship paradox proof

- Consider the event:
  $Event_k$=a node with $d_u = k$ attaches to one with $d_v = k$
- The expected numbers of nodes with degree k is Np_k and so:

$$P(Event_k) = \frac{k}{2M-1} Np_k \approx \frac{kp_k}{d^{\text{ave}}}$$

where $d^{\text{ave}} = 2M/N$ is the average degree due to the handshaking theorem (we ignored the -1 in the denominator, assuming $M \gg 1$). It follows

$$\mathbb{E}[d_v | d_u = k, (u,v) \in \mathcal{E}] = \sum_k kP(Event_k) = \frac{1}{d^{\text{ave}}} \sum_{k=1}^{N} k^2 p_k$$

$$= \frac{1}{d^{\text{ave}}} (\sigma_d^2 + (d^{\text{ave}})^2) = \frac{\overbrace{\sigma_d^2}^{>0}}{d^{\text{ave}}} + d^{\text{ave}} > d^{\text{ave}}$$

Random graphs with given expected degree

- Can we modify $\mathcal{G}(N,p)$ to impose a degree distribution different from the Binomial?
- Yes, by appropriately selecting the edge probabilities p_{uv} insteand of assuming that it is a constant p
- This is named the *Chung-Lu model* and the choice of p_{uv} that is used is as follows:

$$p_{uv} = \begin{cases} \frac{c_u c_v}{2M} & u \neq v \\ \frac{c_u^2}{4M} & \text{else} \end{cases}$$

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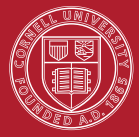
Conclusions



- We have surveyed a couple of important stochastic models for graphs
- A benchmark is the Poisson random graph
- A case that allows for more realism is the configuration model
- Next time we will consider a few more models and compare their average features with real networks